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Heterotic Anomaly Cancellation in Five Dimensions

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Abstract

We study the constraints on five–dimensional $\mathcal{N} = 1$ heterotic M–theory imposed by a consistent anomaly–free coupling of bulk and boundary theory. This requires analyzing the cancellation of triangle gauge anomalies on the four–dimensional orbifold planes due to anomaly inflow from the bulk. We find that the semi–simple part of the orbifold gauge groups and certain $U(1)$ symmetries have to be free of quantum anomalies. In addition there can be several anomalous $U(1)$ symmetries on each orbifold plane whose anomalies are cancelled by a non–trivial variation of the bulk vector fields. The mixed $U(1)$ non–abelian anomaly is universal and there is at most one $U(1)$ symmetry with such an anomaly on each plane. In an alternative approach, we also analyze the coupling of five–dimensional gauged supergravity to orbifold gauge theories. We find a somewhat generalized structure of anomaly cancellation in this case which allows, for example, non–universal mixed $U(1)$ gauge anomalies. Anomaly cancellation from the perspective of four–dimensional $\mathcal{N} = 1$ effective actions obtained from $E_8 \times E_8$ heterotic string– or M–theory by reduction on a Calabi–Yau three–fold is studied as well. The results are consistent with the ones found for five–dimensional heterotic M–theory. Finally, we consider some related issues of phenomenological interest such as model building with anomalous $U(1)$ symmetries, Fayet–Illiopoulos terms and threshold corrections to gauge kinetic functions.

1 Introduction

Green–Schwarz type anomaly cancellation of gauge and gravitational quantum anomalies [1] is one of the key ingredients of string theory. Its role became even more pronounced in the wider context of M–theory, particularly in situations where branes are coupled to higher–dimensional bulk theories. In those cases, quantum anomalies that may arise on the brane worldvolume are cancelled by an anomalous variation of the bulk theory which is of Green–Schwarz type. This mechanism is also referred to as anomaly inflow [2, 3] from the bulk into the brane. A prominent example is given by the M–theory five–brane. Once coupled to 11–dimensional supergravity, its quantum anomaly is cancelled by anomaly inflow induced by the variation of a Green–Schwarz term that has to be added to 11–dimensional supergravity [4]. Another example, that is of particular relevance for the present paper is provided by the Hořava–Witten construction [5, 6] of strongly coupled $E_8 \times E_8$ heterotic string. In this construction, 11–dimensional supergravity on the orbifold S^1/Z_2 is considered. The quantum anomalies of the E_8 gauge fields residing on the two 10–dimensional orbifold fixed planes as well as gravitational anomalies arising on those planes are cancelled by anomaly inflow from 11–dimensional supergravity.

Green–Schwarz anomaly cancellation mechanisms are particularly relevant from two different perspectives. First, they provide a tool to construct consistent anomaly–free theories, particularly those in which branes are coupled to bulk theories. In fact, anomaly cancellation has been the main guidance in the construction of 11–dimensional supergravity on the orbifold S^1/Z_2 [5, 6] as well as on other orbifolds [7, 8, 9, 10, 11]. Secondly, Green–Schwarz anomaly cancellation or certain remnants thereof arising in low–energy effective theories associated with string– or M–theory compactifications are of considerable phenomenological importance. Correspondingly, such low–energy cancellation mechanisms, that is the "anomalous" $U(1)$ symmetries that typically arise in this context, have been investigated early on, particularly in heterotic theories [12, 13, 14, 15, 16, 17, 18]. A more recent analysis for string orbifolds can be found in Ref. [19].

Aspects of anomalous $U(1)$ symmetries in the context of heterotic M–theory have been addressed in Ref. [20, 21, 22]. Particularly, Ref. [21] contains an interesting discussion of some of the issues that become apparent in the strongly coupled limit. Recently, there is also considerable interest in anomaly cancellation of low–energy models from type I string theory [23, 24, 25, 26, 27] and its relation to heterotic models [25].

The purpose of the present paper is to investigate systematically low–energy anomaly cancellation in the context of the $E_8 \times E_8$ heterotic string– or M–theory. That is, we will determine the structure of anomalous $U(1)$ symmetries both from the viewpoint of the four– and five–dimensional effective actions. This will be done in a three–fold way. In a first step, we will investigate anomaly cancellation in five–dimensional heterotic M–theory [28, 29, 30]. This theory is obtained from its

11–dimensional counterpart (that is, the Hořava–Witten construction) by reduction on a Calabi–Yau three–fold with a non–zero mode of the antisymmetric tensor field and general vector bundles. Properties of such non–standard embedding vacua with general bundles, within the context of the strongly coupled heterotic string, have been considered in Ref. [31, 32, 21, 33]. The five–dimensional effective actions associated to such vacua have been analyzed in Ref. [33]. These theories consist of a five–dimensional $\mathcal{N} = 1$ gauged bulk supergravity coupled to two four–dimensional orbifold planes that carry $\mathcal{N} = 1$ gauge theories. Within this setting we will study the cancellation of quantum (gauge) anomalies on the orbifold planes induced by anomaly inflow from the five–dimensional bulk supergravity theory. In other word, we are analyzing, within the framework of heterotic M–theory, the set of four–dimensional gauge theories that can be consistently coupled to gauged five–dimensional $\mathcal{N} = 1$ supergravity.

In a second step, we will study a similar question in a somewhat more general context. We will consider plain 11–dimensional supergravity reduced on a Calabi–Yau three fold in the presence of an internal non–zero mode of the antisymmetric tensor field. The resulting five–dimensional theory is again a gauged $\mathcal{N} = 1$ supergravity similar to the bulk part of five–dimensional heterotic M–theory. Then we shall consider this five–dimensional theory on an S^1/Z_2 orbifold, viewed as a background solution to the five-dimensional supergravity equations. By the use of anomaly cancellation, we shall investigate which four–dimensional $\mathcal{N} = 1$ gauge theories (arising as "twisted states" on the orbifold fixed planes) can be consistently coupled to this five–dimensional supergravity theory. Note that in this approach, in contrast to the previous one, the orbifold planes and their particle content are not necessarily inherited from the 11 dimensional Hořava–Witten construction. It is therefore more general and can be expected to lead to a wider class of models.

Both the above approaches are aimed at finding consistent five–dimensional M–theory models coupled to four–dimensional gauge theories by using the constraints arising from anomaly cancellation. Clearly, these models are of considerable phenomenological interest as they may lead to potentially realistic $\mathcal{N} = 1$ supergravity models in four dimensions. Finally, to complement the five–dimensional approaches and to investigate more closely the phenomenological aspects, we will perform a similar analysis directly for the four–dimensional effective actions associated with the $E_8 \times E_8$ heterotic string– or M–theory reduced on S^1 times Calabi–Yau three–folds. That is, we will analyze the structure of anomalous $U(1)$ symmetries from a four–dimensional viewpoint.

Given the considerable literature already existing on anomalous $U(1)$ symmetries in heterotic theories, what is the motivation for our present study? First of all, we are interested in the new five–dimensional aspects of the problem such as anomaly inflow. Moreover, the five–dimensional picture appears to immediately run into conflict with certain assumptions that are usually made about anomalous $U(1)$ symmetries in the $E_8 \times E_8$ heterotic theory. Let us cite some of these standard assumptions. The starting point is the assertion that the various $U(1)$ quantum anomalies are cancelled by a shift of the dilaton S. Due to its universal coupling, there is at most one anomalous $U(1)$ symmetry which has the same mixed gauge anomaly coefficients for all gauge factor (and, in fact, for gravity, as well) in the observable and the hidden sector. This implies that there have to be matter fields charged under this $U(1)$ symmetry in both sectors. In this sense, the anomalous $U(1)$ symmetry "extends" over the observable and the hidden sector. How can such a $U(1)$ symmetry arise in a five–dimensional theory in which the hidden and the observable sector are confined to four–dimensional brane worldvolumes that are spatially separated by a "gravity only" bulk? In this paper, we attempt to resolve this puzzle as well as related ones, thereby establishing a consistent picture of $E_8 \times E_8$ low–energy anomaly cancellation.

In the next section we begin by deriving the effective five–dimensional action of heterotic M–theory. While some of the results have already been obtained in the literature [28, 29, 30, 33], we will focus on those aspects that are particularly relevant for our purposes. Specifically, we will derive all parts of the effective action that are related to bulk antisymmetric tensor fields. Care will be taken, in this derivation, to incorporate the most general structure of gauge fields on the orbifold planes. As a result, we obtain the most general form of bulk Chern–Simons terms as well as Bianchi identities. The latter controls the coupling between the bulk antisymmetric tensor fields and the gauge fields on the orbifold planes. We will find it useful to split the associated low–energy gauge groups $\mathcal{G}_n$, where $n = 1, 2$ numbers the orbifold planes, into two parts, that is, $\mathcal{G}_n = \mathcal{H}_n \times \mathcal{J}_n$. The first part, $\mathcal{H}_n$ contains the semi–simple part of the gauge group and certain $U(1)$ factors. We will describe these $U(1)$ factors as being of type II. They are characterized by the property that the $U(1)$ factor is not part of the internal bundle structure group. On the other hand, $\mathcal{J}_n$ contains $U(1)$ factors that we shall describe as being of type I. Their defining property is that the $U(1)$ symmetry is part of the internal bundle structure group. This distinction between two types of $U(1)$ symmetries is not new [12, 17]. However, it is frequently not taken into account explicitly.

In section 3, we shall study the classical variation of the five–dimensional action under orbifold gauge symmetries. This classical variation arises due to a non–trivial gauge transformation law of the bulk vector fields residing in the $\mathcal{N} = 1$ gravity and vector multiplets. An important role is also played by a certain bulk Chern–Simons term that originates from the internal non–zero mode of the anti–symmetric tensor field and is closely related to the gauging of the five–dimensional supergravity theory [28, 30]. Carrying out an index theorem calculation [34, 35], we check explicitly that this classical variation cancels the quantum variation on the orbifold planes due to triangle diagrams. As a result, we obtain the triangle anomaly coefficients in terms of topological data of the underlying Calabi–Yau compactification. This implies the following general structure of anomalies. First of all, anomaly cancellation works independently on each orbifold plane due to anomaly inflow from the bulk. Further, the $\mathcal{H}_n$ parts of the gauge groups are quantum anomaly–free, that is, cubic anomalies within $\mathcal{H}_n$ and mixed gravitational anomalies of $\mathcal{H}_n$ gauge fields vanish. For the semi-simple part of $\mathcal{H}_n$ this result is well–known [34]. In addition, we conclude that the type II $U(1)$ symmetries in $\mathcal{H}_n$ are anomaly–free in above sense. The remaining $U(1)$ symmetries of type I in $\mathcal{J}_n$ may have quantum anomalies that are cancelled by anomaly inflow. The associated anomaly coefficients for the hidden and the observable sector are generically unrelated. On each orbifold plane, the anomaly coefficients for the mixed $U(1)$ gauge anomalies (diagrams with one type I $U(1)$ gauge field and two gauge fields from $\mathcal{H}_n$), the mixed $U(1)$ gravitational anomalies (diagrams with one type I $U(1)$ gauge field and two gravitons) and the cubic $U(1)$ anomalies (diagrams with three type I $U(1)$ gauge fields) can be non–zero and are likewise generically unrelated. However, the mixed $U(1)$ gauge anomaly is the same for each factor within $\mathcal{H}_n$ and is in this sense universal. This implies that, in each sector, there is at most one type I $U(1)$ symmetry with all three types of anomalies non–vanishing. In addition, there can be other type I $U(1)$ symmetries with vanishing mixed gauge anomaly but non–vanishing gravitational and cubic anomaly. We stress again that those statements apply independently two both sectors. There could, for example, be two "anomalous" $U(1)$ symmetries, one in each sector, both with mixed gauge anomalies but unrelated anomaly coefficients. As another example, there could be an anomalous $U(1)$ symmetry in the observable sector while the hidden sector is non–anomalous.

In section 4, we will study five–dimensional anomaly cancellation in the somewhat more general context described above. While, of course, we reproduce all the models that were found in the context of five–dimensional heterotic M–theory, there are also some generalizations. Most notably, the total size of the orbifold gauge groups $\mathcal{G}_n$ is not restricted to fit into E_8 and the mixed $U(1)$ gauge anomaly can also depend on the factor in $\mathcal{H}_n$, that is, it can be non–universal. Although those generalizations can be realized by coupling four–dimensional gauge theories to a special version of five–dimensional supergravity obtained from M–theory, it is not clear whether they are actually part of M–theory. One may attempt to realize such models in the framework of heterotic M–theory on Calabi–Yau three–folds with five–branes [33, 36, 37, 38]. Although this is of considerable phenomenological interest we will not explicitly address this issue here.

In section 5, we derive the relevant parts of the four–dimensional effective actions associated with $E_8 \times E_8$ heterotic string– or M–theory on S^1 times a Calabi–Yau three–fold. Again, we study the classical variation of this action to find the pattern of anomalous $U(1)$ symmetries. As required on general grounds, the results turn out to be consistent with the ones obtained for five–dimensional heterotic M–theory. We also gain some more insight into the four–dimensional mechanism that leads to Green–Schwarz cancellation. In the standard gauge kinetic functions for $\mathcal{H}_n$ given by $f_n = S \mp \epsilon_S \beta_i T^i$ it is a shift of the T^i moduli rather than the dilaton S that leads to anomaly cancellation. This resolves the puzzle raised above, since the standard assumptions about anomalous $U(1)$ symmetries are based on a cancellation induced by a dilaton shift. In fact, the confusion is precisely one between the type I and type II $U(1)$ symmetries. Anomalous type

II (as well as presumably type I) $U(1)$ symmetries arise in models originating from the $SO(32)$ heterotic theory on a Calabi-Yau three–fold. In those cases, the type II anomaly cancellation works via a shift of the dilaton and, hence, the $U(1)$ symmetry should have the universal properties that are commonly assumed. However, in models originating from the $E_8 \times E_8$ heterotic theory on a Calabi–Yau three–fold, we have seen that type II $U(1)$ symmetries are alway non–anomalous. The basic reason for this difference [17] is that $SO(32)$ has an independent fourth order invariant while E_8 has not. Hence, applying the type II anomaly cancellation patterns to $E_8 \times E_8$ models is not appropriate as type II symmetries are always anomaly–free in such models. The type I $U(1)$ symmetries that may be anomalous in $E_8 \times E_8$ models, however, have a very different pattern of anomaly cancellation, as described above.

Finally, in section 6, we shall discuss some phenomenological consequences of our results. We start with some general remarks about possible applications to model–building with anomalous $U(1)$ symmetries. We also point out that, within special classes of compactifications, our results can be used to further constrain the structure of the anomaly coefficients. As an example, we discuss symmetric vacua which have recently been constructed [39]. An important issue related to anomalous $U(1)$ symmetries are Fayet–Illiopoulos terms. We show that such terms arise from the dilaton part of the Kähler potential, despite the different rôle of the dilaton. In accordance with the required independence of the two sectors, we may have two FI terms, one in each sector. They are proportional to the respective mixed $U(1)$ gauge anomaly coefficients. Finally, we also analyze the gauge kinetic functions of the type I $U(1)$ symmetries, which turn out to be more complicated than the standard ones for the $\mathcal{H}_n$ parts of the gauge group.

Before proceeding to the details of the M-theory anomaly cancellation, we would like to raise the general question of whether the present results support the idea that M-theory is simply $D = 11$ supergravity, properly quantized (and thus including its essential brane sectors). One indication that may go in this direction is the analysis of M-theory/heterotic duality given in Ref. [40], taken in the context of $D = 11$ supergravity with branes wrapped and stacked around cycles of Calabi-Yau manifolds. In this analysis, the Hořava-Witten orbifold S^1/Z_2 appears in a certain singular limit of 11–dimensional supergravity on a $K3$ surface. The $E_8 \times E_8$ gauge fields on the orbifold fixed points arise from membranes wrapping spheres within $K3$ that collapse in the singular limit. A central question here is whether the full gauge group structure in the resulting $D = 5$ theory can be viewed as arising in a similar way. The results of Ref. [40] would seem to suggest that this is indeed the case for the analysis of sections three and five, that is, for all five–dimensional theories obtained by a reduction of 11–dimensional Hořava-Witten theory. The details of how that works will have to be addressed in a future publication.

2 The five–dimensional effective action of heterotic M–theory

In this section we shall derive the five–dimensional effective action of heterotic M–theory. This action has first been given in Ref. [28, 29, 30] for the case of a standard embedding. Some of the generalizations that emerge upon allowing general vector bundles in the vacuum have been given in Ref. [33]. However, certain aspects, particularly some relevant for the question of anomaly cancellation, have not been explicitly addressed in the literature. For the sake of clarity, we will therefore present the reduction in a systematic way, focusing on issues related to anomaly cancellation. This implies, in particular, that we need to consider the most general structure for the E_8 vector bundles.

2.1 The Hořava–Witten action in $D = 11$

The starting point of our reduction is the action for 11–dimensional supergravity on an S^1/Z_2 orbifold, due to Hořava and Witten [5, 6]. More precisely, we consider 11–dimensional supergravity on the space $M_{11} = S^1/Z_2 \times M_{10}$ where M_{10} is a smooth 10–dimensional manifold. The orbifold coordinate is denoted by y throughout the paper and is taken to be in the range $y \in [-\pi\rho, \pi\rho]$. The Z_2 symmetry acts as $y \to -y$ (or $y \to -y + 2a$ for reflections around points other than $y = 0$). Hence, selecting the reflection point $y = 0$ (with a conjugate reflection point at the reidentification point $y = \pm\pi\rho$) there exist two 10–dimensional fixed hyperplanes, $M_{10}^{(1)}$ at $y = y_1 \equiv 0$ and $M_{10}^{(2)}$ at $y = y_2 \equiv \pi\rho$. The bulk action is given by 11–dimensional supergravity with the metric g and the three–index antisymmetric tensor field C as the bosonic fields. In the bulk, the field strength G of C is given by $G = dC$. This bulk theory is coupled to two 10–dimensional E_8 super–Yang–Mills multiplets each residing on one of the orbifold fixed planes. We denote the corresponding gauge fields by A_n and their field strengths by F_n, where $n = 1, 2$. Generators T^a are always normalized such that $\mathrm{tr}(T^a T^b) = \delta^{ab}$ where, as usual, tr is defined to be 1/30 of the trace in the adjoint.

The action for this theory can be organized as an expansion in powers of $\kappa^{2/3}$, where κ is the 11–dimensional Newton constant. In order to simplify the notation, it is helpful to introduce the specific combination [1]

$$\lambda = \frac{c}{2\sqrt{2}\pi}\left(\frac{\kappa}{4\pi}\right)^{2/3} \ . \tag{2.1}$$

Then the action has the structure

$$S_{11} = S_0 + S_1 + S_2 \ , \tag{2.2}$$

where the subscripts on the right hand side refer to the order in λ. In the following, we shall focus on the bosonic part of this action, which will be sufficient for the purpose of the present paper. To

[1] In Hořava and Witten's original formulation of the theory the coefficient c was determined to be $c = 1$. Subsequently, it was found in Ref. [41, 42] that $c = 2^{-1/3}$. In this paper, we will use $c = 2^{-1/3}$.

zeroth order, we have the action of 11–dimensional supergravity

$$2\kappa^2 S_0 = -\int_{M_{11}} \left[\sqrt{-g}\, R + G \wedge *G + \frac{\sqrt{2}}{3} C \wedge G \wedge G \right] . \tag{2.3}$$

The Yang–Mills theories on the orbifold fixed planes appear at first order in λ. They are specified by

$$2\kappa^2 S_1 = -\frac{\lambda}{\sqrt{2}} \sum_{n=1}^{2} \int_{M_{10}^{(n)}} \sqrt{-g} \left[\mathrm{tr} F_n^2 - \frac{1}{2} \mathrm{tr} R^2 \right] . \tag{2.4}$$

At order λ^2, a Green–Schwarz term [4] and associated R^4 terms [43, 44] arise in the bulk. Here, we only need the Green–Schwarz term which is given by

$$2\kappa^2 S_2 = \frac{\sqrt{2}}{3} \lambda^2 \int_{M_{11}} C \wedge X_8 \tag{2.5}$$

where the anomaly polynomial X_8 reads [2]

$$X_8 = -\frac{1}{8} \mathrm{tr} R^4 + \frac{1}{32} (\mathrm{tr} R^2)^2 . \tag{2.6}$$

Finally, the Bianchi identity for G has to be modified by source terms supported on the orbifold planes that appear at order λ. This modified Bianchi identity reads

$$dG = -\lambda \sum_{n=1}^{2} J_n \wedge dy\, \delta(y - y_n) . \tag{2.7}$$

The sources J_n depend on the gauge field and the curvature on the orbifold planes and are explicitly given by

$$J_n = \mathrm{tr} F_n^2 - \frac{1}{2} \mathrm{tr} R^2 \,|_{y=y_n} . \tag{2.8}$$

2.2 Vacuum configuration

We would like to consider the above theory on a space–time M_{11} with structure

$$M_{11} = X \times S^1/Z_2 \times M_4 \tag{2.9}$$

where X is a Calabi–Yau three–fold. Eventually, we will be interested in the five–dimensional theory on $M_5 = S^1/Z_2 \times M_4$ that is obtained by compactification on the Calabi–Yau space X. Let us describe the background configurations adequate for such a compactification [28, 30].

In the bulk, we have a Calabi–Yau background metric $\bar{g}$ and an associated curvature two–form $\bar{R}$ related to the tangent bundle TX of the Calabi–Yau space X. Here and in the following we use the bar to indicate fields with components exclusively in the internal space X.

[2]Here and in the following we will frequently omit the wedge symbol $\wedge$ in writing wedge products in order to simplify the notation.

Furthermore, we should specify the internal parts of the E_8 gauge fields. Rather than making any specific choice, such as the standard embedding of the spin connection into the gauge group, we would like to consider the general situation. That is, we would like to cover all choices of internal gauge fields compatible with the consistency requirements imposed by the theory. Allowing for this most general situation is, of course, crucial in our context since those gauge fields play a key role in anomaly cancellation. We start with two E_8 vector bundles V_1 and V_2 which are restricted so as to preserve supersymmetry, possessing thus a property that is known formally as semi-stability (see, *e.g.* [17]). Integrating the Bianchi identity (2.7) over a five–cycle consisting of a four–cycle within the Calabi–Yau space times the orbifold leads to the familiar constraint

$$\mathrm{ch}_2(V_1) + \mathrm{ch}_2(V_2) = \mathrm{ch}_2(TX) \tag{2.10}$$

on those bundles. Here ch_2 is the second Chern character. The field strengths associated to the bundles V_n are denoted by $\bar{F}_n$, where $n = 1, 2$. We also introduce the internal parts $\bar{J}_n$ of the sources in the Bianchi identity (2.8) by

$$\bar{J}_n = \mathrm{tr}\bar{F}_n^2 - \frac{1}{2}\mathrm{tr}\bar{R}^2 \left.\right|_{y=y_n} . \tag{2.11}$$

The consistency relation (2.10) can then be written in the equivalent form

$$\int_{C_{4i}} (\bar{J}_1 + \bar{J}_2) = 0 \tag{2.12}$$

where $\{C_{4i}\}_{i=1,\ldots,h^{1,1}}$ is a basis of Calabi–Yau four–cycles. Furthermore, it is useful to introduce the topological indices β_i and γ_i by

$$\begin{aligned} \beta_i &= -\frac{1}{8\pi^2}\int_{C_{4i}} \bar{J}_1 \\ &= -\frac{1}{8\pi^2}\int_{C_{4i}} \left[\mathrm{tr}\bar{F}_1^2 - \frac{1}{2}\mathrm{tr}\bar{R}^2\right] \qquad (2.13) \\ \gamma_i &= -\frac{1}{8\pi^2}\int_{C_{4i}} \mathrm{tr}\bar{R}^2 . \qquad (2.14) \end{aligned}$$

It is important to be somewhat more specific about the group structure of our vector bundles. Following Ref. [12, 17] we write the structure groups $\bar{\mathcal{G}}_n$ of the bundles V_n as products

$$\bar{\mathcal{G}}_n = \bar{\mathcal{H}}_n \times \mathcal{J}_n . \tag{2.15}$$

Here $\bar{\mathcal{H}}_n$ is the semi–simple part and $\mathcal{J}_n$ contains the $U(1)$ factors. We also allow for Wilson lines which can be thought of as discrete parts of the gauge bundle. We choose the associated discrete structure group to be part of $\mathcal{H}_n$. Correspondingly, we write the bundles and the gauge field strengths as

$$V_n = W_n \oplus \bigoplus_a \mathcal{L}_n^a , \qquad \bar{F}_n = \bar{f}_n + \sum_a \bar{\mathcal{F}}_n^a Q_n^a \tag{2.16}$$

where the bundles W_n and associated gauge fields $\bar{f}_n$ correspond to the semi–simple part (and possible Wilson lines) while the line bundles $\mathcal{L}_n^a$ with $U(1)$ gauge fields $\bar{\mathcal{F}}_n^a$ correspond to the various $U(1)$ factors. The associated $U(1)$ generators are denoted by Q_n^a. Here and in the following we label these $U(1)$ factors by indices $a, b, \ldots$. In order to satisfy the field equations, the $U(1)$ field strengths have to be harmonic two–forms. Hence they can be written as

$$\bar{\mathcal{F}}_n^a = v^{-1/3} \eta_{na}^i \Omega_i \ . \tag{2.17}$$

Here $\{\Omega_i\}_{i=1,\ldots,h^{1,1}}$ is a basis of harmonic $(1,1)$ forms on the Calabi–Yau space dual to the basis of four–cycles mentioned above. For dimensional reasons, we have also included a power of the Calabi–Yau coordinate volume v in the above definition. As usual, the coefficients η_{na}^i in this expansion are quantized in suitable units. We have mentioned above that the vector bundles V_n have to be semi–stable so as to preserve supersymmetry. In particular, this implies for the line bundles $\mathcal{L}_n^a$ that

$$\Omega \wedge \Omega \wedge c_1(\mathcal{L}_n^a) = 0 \tag{2.18}$$

where Ω is the Kähler form. Clearly, this condition cannot be satisfied for $h^{1,1} = 1$. Hence, only for Calabi–Yau spaces with $h^{1,1} > 1$ can the $U(1)$ factors introduced above occur. The conditions (2.18) in fact constitute restrictions on the Calabi–Yau Kähler moduli space [12]. More explicitly, this can be seen by writing the Kähler form as $\Omega = a^i \Omega_i$ with the $(1,1)$ moduli a^i. Then, inserting this into Eq. (2.18) along with the expansion (2.17) of the $U(1)$ gauge fields leads to

$$d_{ijk} \eta_{na}^i a^j a^k = 0 \ . \tag{2.19}$$

The constants d_{ijk} are the Calabi–Yau triple intersection numbers defined as

$$d_{ijk} = \int_X \Omega_i \wedge \Omega_j \wedge \Omega_k \ . \tag{2.20}$$

In order to satisfy the equations (2.19) we have to adjust the Kähler moduli since the coefficients η_{na}^i, once chosen, are fixed due to the quantization rule.

Finally, we have to specify the background value of the antisymmetric tensor field strength G. In fact, a non–vanishing internal G is forced upon us by the Bianchi identity (2.7) taking into account the generically non–trivial internal sources $\bar{J}_n$. This background value for G, also called a non–zero mode, plays an important role in the derivation of the five–dimensional effective theory, as has been demonstrated in Ref. [28, 30]. As we will see, taking into account this non–zero mode is also crucial in order to understand anomaly cancellation in five dimensions. Solving the Bianchi identity (2.7) subject to the sources (2.11) along with the equation of motion for G leads to the following expression for the non–zero mode

$$\bar{G} = 4\pi^2 \lambda \, \beta_i \nu^i \, \epsilon(y) \ . \tag{2.21}$$

Here $\{\nu^i\}_{i=1,\dots,h^{1,1}}$ forms a basis of harmonic $(2,2)$ forms on the Calabi–Yau space dual to the basis $\{\Omega_i\}$ of harmonic $(1,1)$ forms. The step–function $\epsilon(y)$ is defined to be $+1$ for $y \geq 0$ and -1 otherwise.

Note that the presence of the step–function in this background value for G breaks the five-dimensional translation invariance at the selected orbifold points $y = 0$, $y = \pm\pi\rho$. Accordingly, the $D = 5$ field theory will be found to have extended–objects at the locations of the step–function jumps. At the present time, however, we see only a hint of this structure in (2.21), which constitutes a generalization of the usual kind of Kaluza-Klein ansatz for field configurations in the internal dimensions. Note finally that although (2.21) breaks the $D = 5$ translation invariance, it actually *restores* the Z_2 symmetry $y \to -y$ of the $D = 11$ theory, which would otherwise be broken if the step–function were not present in (2.21), since G is a Z_2 odd quantity.

2.3 Zero modes

We would now like to summarize the structure of zero modes arising for backgrounds of the above type. Let us start with bulk zero modes. As we will see, for the discussion of five–dimensional anomalies, we can focus on the zero modes of the antisymmetric tensor field related to the constant Calabi–Yau mode and the $(1,1)$ sector. In addition, we will consider the bulk modes needed to complete $\mathcal{N} = 1$ multiplets in five dimensions.

The Calabi–Yau breathing mode V is defined as

$$V = \frac{1}{v}\int_X \sqrt{g}\ . \tag{2.22}$$

In the previous subsection we have already introduced the $(1,1)$ moduli a^i, where $i = 1, \dots, h^{1,1}$. Those are, however, not independent from the volume modulus V. To remove the redundancy, it is useful to define the "shape moduli" b^i by

$$b^i = V^{1/3} a^i\ . \tag{2.23}$$

Those $h^{1,1}$ quantities are subject to one constraint. Hence they represent only $h^{1,1} - 1$ independent degrees of freedom. Neglecting contributions from harmonic $(2,1)$ forms, the antisymmetric tensor three–form and its field strength can be written as

$$C = \bar{C} + \tilde{C} + \xi\Omega_3 + \xi^*\Omega_3^* + \mathcal{B}^i \wedge \Omega_i \tag{2.24}$$

$$G = \bar{G} + \tilde{G} + X \wedge \Omega_3 + X^* \wedge \Omega_3^* + \mathcal{D}^i \wedge \Omega_i\ . \tag{2.25}$$

Here $\bar{C}$ and $\bar{G}$ represent the non–zero mode background specified in the previous subsection. The field $\tilde{C}$, with field strength $\tilde{G}$, represent a three–index antisymmetric tensor field in the external five dimensions that is associated to the constant mode on the Calabi–Yau space. The complex scalar ξ and its field strength X arise from the harmonic $(3,0)$ form Ω_3. Furthermore, we have $h^{1,1}$ gauge fields $\mathcal{B}^i$ with field strengths $\mathcal{D}^i$ related to the harmonic $(1,1)$ forms Ω_i.

In five dimensions, these zero modes give rise to the following multiplets. The five–dimensional metric along with a certain linear combination of the vectors $\mathcal{B}^i$ form the bosonic part of the $\mathcal{N}=1$ gravity multiplet. The remaining $h^{1,1}-1$ vectors $\mathcal{B}^i$ together with the scalars b^i represent $h^{1,1}-1$ vector multiplets. Finally, the volume modulus V, the complex scalar ξ and the dual of the three–form $\tilde{C}$ (which is a scalar in five dimensions) form the universal hypermultiplet.

Next we should consider the zero modes arising on the orbifold planes. After compactification, these planes become four–dimensional hyperplanes $M_4^{(n)}$ located at $y = y_n$ in the five–dimensional space. Recall, that the internal gauge groups have a product structure $\bar{\mathcal{G}}_n = \bar{\mathcal{H}}_n \times \mathcal{J}_n$ where $\bar{\mathcal{H}}_n$ are the semi–simple parts and $\mathcal{J}_n$ contains the $U(1)$ factors. The surviving low–energy groups $\mathcal{G}_n$ are the commutants of these internal groups within E_8. Their structure is given by

$$\mathcal{G}_n = \mathcal{H}_n \times \mathcal{J}_n \; . \tag{2.26}$$

where $n = 1, 2$. Note that the $U(1)$ factors commute with themselves. Hence $\mathcal{J}_n$ is part of the low–energy gauge groups as well. We will call such $U(1)$ factors in $\mathcal{J}_n$ that have a counterpart in the internal structure group $U(1)$ symmetries of type I, or $U_I(1)$ in short. The remaining parts of the low–energy groups contain the non–abelian factors and are denoted by $\mathcal{H}_n$. Note, however, that the groups $\mathcal{H}_n$ do not necessarily have to be semi–simple. The low–energy group may contain $U(1)$ factors other than the type I ones mentioned above, which would then be contained in $\mathcal{H}_n$. We will call these type II $U(1)$ symmetries, or $U_{II}(1)$ in short. It is clear, from their relation to the internal structure groups, that type I and type II $U(1)$ fields are of very different nature. This will become apparent in the discussion of anomaly cancellations. We also introduce gauge fields $\tilde{A}_n$ with field strengths $\tilde{F}_n$ for the group $\mathcal{H}_n$ and $U(1)$ gauge fields $\tilde{\mathcal{A}}_n^a$ with field strengths $\tilde{\mathcal{F}}_n^a$ for the type I $U(1)$ factors in $\mathcal{J}_n$.

To discuss anomalies, we also need some information about the $\mathcal{N}=1$ chiral multiplets on the orbifold planes. As usual we decompose the adjoint of E_8 under the subgroup $\bar{\mathcal{G}}_n \times \mathcal{G}_n$ as

$$\mathbf{248}_{E_8} \to \bigoplus_r (\bar{L}_n^r, L_n^r) \tag{2.27}$$

where the sum runs over all representations $(\bar{L}_n^r, L_n^r)$ of $\bar{\mathcal{G}}_n \times \mathcal{G}_n$. Then, in general, we expect chiral $\mathcal{N}=1$ multiplets in all representation L_n^r of the external gauge group $\mathcal{G}_n$ that appear in this decomposition. Fortunately, all we need to know in this context is the chiral asymmetry of such multiplets in L_n^r which we denote by N_n^r. From the index theorem this asymmetry is given by

$$N_n^r = \frac{1}{6}\frac{1}{(2\pi)^3}\int_X \left[\mathrm{tr}_{\bar{L}_n^r} \bar{F}_n^3 - \frac{1}{8}\mathrm{tr}_{\bar{L}_n^r} \bar{F}_n \mathrm{tr}\bar{R}^2 \right] \; . \tag{2.28}$$

where $\mathrm{tr}_{\bar{L}_n^r}$ denotes the trace taken in the representation $\bar{L}_n^r$ of the internal gauge group $\bar{\mathcal{G}}_n$. Recall also that $\bar{F}_n$ and $\bar{R}$ are the internal gauge field and curvature backgrounds.

2.4 The five–dimensional effective action

We are now ready to summarize the parts of the five–dimensional effective action that are essential for our purpose. Starting from the 11–dimensional theory and using the backgrounds and the zero modes introduced above one finds

$$S_5 = S_{\rm kin} + S_{\rm top} + S_{\rm bound} \ . \tag{2.29}$$

Here $S_{\rm kin}$ contains the kinetic terms of the antisymmetric tensor fields and is given by

$$2\kappa_5^2\, S_{\rm kin} = -\int_{M_5} \left[2G_{ij}\mathcal{D}^i \wedge *\mathcal{D}^j + 2V^{-1}X \wedge *X^* + V^2 G \wedge *G\right] \ . \tag{2.30}$$

We have omitted the kinetic terms of all other moduli since they will not be relevant for the subsequent discussion. The Kähler moduli space metric G_{ij} will not be explicitly needed here. The topological part of the action reads

$$\begin{aligned}2\kappa_5^2\, S_{\rm top} = -\sqrt{2}\int_{M_5} \Bigg[\frac{\pi^2\lambda^2}{6v^{2/3}}\,\gamma_i\,\mathcal{B}^i \wedge \mathrm{tr}R^2 - \frac{8\pi^2\lambda}{v^{2/3}}\epsilon(y)\,\beta_i\,\mathcal{B}^i \wedge G + \frac{1}{3}d_{ijk}\mathcal{B}^i\mathcal{D}^j\mathcal{D}^k \\ +i(\xi G \wedge X^* - \xi^* G \wedge X)\Bigg] \ . \end{aligned}\tag{2.31}$$

For the boundary part we have

$$S_{\rm bound} = -\frac{1}{4g_0^2}\sum_{n=1}^{2}\int_{M_4^{(n)}}\sqrt{-g}\left[V\mathrm{tr}F_n^2 + V\sum_a(\mathcal{F}_n^a)^2 + (\text{matter})\right] \ . \tag{2.32}$$

Furthermore, the above action has to be supplemented with the following Bianchi identities

$$dG = -\lambda\sum_{n=1}^{2} J_n \wedge dy\,\delta(y-y_n) \tag{2.33}$$

$$dX = (\text{matter}) \tag{2.34}$$

$$d\mathcal{D}^i = -2v^{-1/3}\lambda\sum_{n=1}^{2}\eta^i_{na}\mathcal{F}_n^a \wedge dy\,\delta(y-y_n) + (\text{matter}) \tag{2.35}$$

where the sources are given by

$$J_n = \mathrm{tr}F_n^2 + \sum_a \mathcal{F}_n^a \wedge \mathcal{F}_n^a - \frac{1}{2}\mathrm{tr}R^2 \ . \tag{2.36}$$

Those Bianchi identities directly descend from the 11–dimensional one, Eq. (2.7). It is also useful to write them in their integrated form

$$
\begin{aligned}
G &= dC - \lambda \sum_{n=1}^{2} w_n \wedge dy \delta(y - y_n) && (2.37) \\
X &= d\xi + (\text{matter}) && (2.38) \\
\mathcal{D}^i &= d\mathcal{B}^i - 2v^{-1/3}\lambda \sum_{n=1}^{2} \eta^i_{na} \mathcal{A}^a_n \wedge dy \delta(y - y_n) + (\text{matter}) && (2.39)
\end{aligned}
$$

with the Chern–Simons form w_n defined by

$$
dw_n = J_n \; . \tag{2.40}
$$

We have also introduced the five–dimensional Newton constant κ_5 and the gauge coupling g_0. They are given in terms of 11–dimensional quantities by

$$
\kappa_5^2 = \frac{\kappa^2}{v} \; , \qquad g_0^2 = \frac{\kappa_5^2}{\sqrt{2}\lambda} \; . \tag{2.41}
$$

A few comments concerning notation are in order. In the previous subsections we have distinguished five–dimensional fields from their 11–dimensional counterparts by a tilde. From now on, we will be working in five dimensions and we shall omit the tilde for notational simplicity. So, for example, C is the five–dimensional three–index antisymmetric tensor field with field strength G. Furthermore, A_n are the gauge fields with gauge group $\mathcal{H}_n$ and field strength F_n, while $\mathcal{F}^a_n$ are the $U_I(1)$ vector fields with associated gauge group $\mathcal{J}_n$. We also recall that d_{ijk} are the Calabi–Yau intersection numbers (2.20) and β_i and γ_i are topological numbers defined in terms of the internal bundles by Eq. (2.13) and (2.14). Although matter fields on the orbifold planes do play an important role in the following their explicit contributions to the action and the Bianchi identities is not essential. Their presence has, however, been indicated in the above action.

It is instructive, for the following, to understand the 11–dimensional origin of some of the above terms. For example, the first term in the topological part of the action is the reduction of the $C \wedge X_8$ term [45] given in Eq. (2.5). All other topological terms originate from the Chern–Simons term CGG of 11–dimensional supergravity. Particularly, the second term in Eq. (2.31) results by taking one of the field strengths G in CGG to be the internal non–zero mode (2.21). This term causes the gauging of the universal hypermultiplet as has been shown in Ref. [28, 30]. It will also be essential to understand anomaly cancellation in the five–dimensional theory. Another essential part in the above action is the right hand side of the Bianchi identity (2.39). The $U_I(1)$ gauge field strengths $\mathcal{F}^a_n$ in this Bianchi identity arise because their associated gauge group $\mathcal{J}_n$ is part of the internal structure group at the same time. In fact, only in this case can one have a non-vanishing result for the trace $\text{tr}F^2$, where one F is taken to have internal indices and the other one to have external indices. It is for this reason, that the $U_I(1)$ fields of type I in $\mathcal{J}_n$ play a special role. Note that the $U_{II}(1)$ fields, contained in the $\mathcal{H}_n$ part of the low-energy gauge group, do not appear in the Bianchi identity (2.39).

3 Anomaly cancellation in five dimensions

We now shall study anomaly cancellation in the five–dimensional action of heterotic M–theory that we have just derived. We expect the anomaly cancellation to rely on an interplay between bulk and orbifold planes in much the same way as in 11 dimensions. More precisely, in the five–dimensional theory, the gauge theories on the orbifold planes might have quantum anomalies at one loop due to the familiar triangle diagrams. These apparent anomalies should then be cancelled by a classical gauge anomaly of the bulk theory supported on those orbifold planes or, in other words, by anomaly inflow from the bulk. Let us work this out in some detail, starting with the anomalous variation of the bulk action.

3.1 Classical variation of the bulk action

An anomalous variation of the bulk action (2.30), (2.31) should be triggered by the Bianchi identities (2.33)–(2.35) as they represent the only way in which the orbifold gauge fields directly communicate with the bulk fields. Clearly, then, the bulk antisymmetric tensor fields are the only bulk fields that potentially transform under the gauge transformations associated to the orbifold gauge fields. As in 11 dimensions, our strategy will be to keep the antisymmetric tensor field strengths invariant so that the kinetic part (2.30) of the action remains inert. This, however, typically implies assigning non–trivial gauge transformations to the antisymmetric tensor fields themselves. In the case at hand, we can see from the Eqs. (2.37)–(2.39) that we need to assign such transformation to the bulk three–form C and the bulk vector fields $\mathcal{B}^i$. Specifically, while C transforms under the $\mathcal{H}_n$ parts of the orbifold gauge groups, the vector fields $\mathcal{B}^i$ transform under the $U_I(1)$ gauge transformations of type I. We note, however, that the bulk action depends on C only through its field strength G. Therefore, the variation of C is not relevant for our purpose. Hence, we can already draw the important conclusion that the $\mathcal{H}_n$ parts of the gauge groups have to be non–anomalous. More precisely, the one–loop anomalies due to $\mathcal{H}_n$ triangle diagrams have to vanish since, as we have just seen, there is no corresponding bulk variation available to cancel them. We will come back to this in a more systematic way later on.

For now, we should be more explicit about the gauge variation of the $U_I(1)$ fields and the transformation of the bulk vector fields that they induce. We denote the transformation parameters of these $U_I(1)$ fields Λ_n^a and write

$$\delta\mathcal{A}_n^a = d\Lambda_n^a \, . \tag{3.1}$$

Then, demanding that the vector field strengths $\mathcal{D}^i$ be invariant, we conclude from Eq. (2.39) that the vector fields $\mathcal{B}^i$ should transform as

$$\delta\mathcal{B}^i = 2v^{-1/3}\lambda \sum_{n=1}^{2} \eta^i_{na}\Lambda^a_n \, dy \, \delta(y-y_n) \; . \tag{3.2}$$

As we have already mentioned, the kinetic part of the bulk action (2.30) remains invariant under this transformation. The variation of the topological part (2.31), however, is non–trivial and consists of terms that are supported purely on the orbifold planes. The latter, of course, is caused by the delta functions in the transformation law (3.2). To further evaluate this anomalous variation, we need to know the values of the antisymmetric tensor field strengths G and $\mathcal{D}^i$ on the orbifold planes $y = y_n$. To be more specific, we need only the components of those fields transverse to the orbifold since that is all that the anomalous variation depends on. The reason for this is the presence of the dy differential in Eq. (3.2) which projects onto those components. Fortunately, those components are Z_2 odd and, hence, their behavior close to the orbifold planes is completely determined by the source terms in the Bianchi identities. Solving the Bianchi identities (2.33) and (2.35) along with the equations of motion for C and $\mathcal{B}^a_n$ leads to

$$\begin{aligned} dy \wedge \mathcal{D}^i \,|_{y=y_n} &= \mp\lambda v^{-1/3}\eta^i_{na}\,\mathcal{F}^a_n \wedge dy & (3.3)\\ dy \wedge G \,|_{y=y_n} &= \mp\frac{\lambda}{2}\, J_n \wedge dy \; . & (3.4)\end{aligned}$$

With these expressions we finally find for the anomalous variation of the bulk action

$$\begin{aligned}\delta_{\rm cl}S_5 = -\frac{c^3}{128\pi^3}\sum_{n=1}^{2}\int_{M_4^{(n)}} \Lambda^a_n \Big[(\mp 2\beta_i\eta^i_{na})\mathrm{tr}F_n^2 - \left(\mp\beta_i - \frac{1}{12}\gamma_i\right)\eta^i_{na}\mathrm{tr}R^2 \\ + \left(\mp 2\beta_i\eta^i_{na}\delta_{bc} + \frac{1}{6\pi^2}d_{ijk}\eta^i_{na}\eta^j_{nb}\eta^k_{nc}\right)\mathcal{F}^b_n\mathcal{F}^c_n\Big] \; . \end{aligned} \tag{3.5}$$

Here and in the following, when we use alternating signs, the upper sign refers to the first orbifold plane, $n = 1$, while the lower sign refers to the second plane, $n = 2$. The above classical variation of the bulk action should be cancelled by quantum anomalies on the orbifold planes that originate from the usual triangle diagrams. That this is possible at all is due to the obvious but nonetheless important fact that the gauge variation of the bulk action consists solely of terms supported on the orbifold planes. From the form of the above variation it is obvious which types of triangle diagram should be responsible for the cancellation. The first term in (3.5) corresponds to a mixed gauge anomaly with one $U_I(1)$ current of type I and two currents from the $\mathcal{H}_n$ parts of the gauge groups. The second term should be cancelled by a mixed gravitational anomaly with one $U_I(1)$ current and two gravity currents. Finally, the structure of the third term corresponds to a cubic anomaly of the $U_I(1)$ gauge fields. In the following subsection we will show explicitly that the anticipated cancellation indeed works.

3.2 Quantum variation on the orbifold planes

Above we have mentioned the types of triangle diagrams whose anomalous variation should be cancelled by (3.5). Clearly, since we have done our reduction from $D = 11$ to $D = 5$ consistently, we expect this cancellation to work. The consistently derived five–dimensional effective theory should be anomaly-free in the same way as the 11 dimensional action. We still find it useful, if only as a cross–check of our result, to verify this explicitly. To do this, we need to know the anomaly coefficients of the various types of triangle diagrams defined as follows

$$\mathcal{C}_n = \sum_r N_n^r \, \mathrm{tr}_{L_n^r}(T_n^3) \tag{3.6a}$$

$$\mathcal{C}_{na} = \sum_r N_n^r \, \mathrm{tr}_{L_n^r}(Q_n^a T_n^2) \tag{3.6b}$$

$$\mathcal{C}_{nab} = \sum_r N_n^r \, \mathrm{tr}_{L_n^r}(Q_n^a Q_n^b T_n) \tag{3.6c}$$

$$\mathcal{C}_{nabc} = \sum_r N_n^r \, \mathrm{tr}_{L_n^r}(Q_n^a Q_n^b Q_n^c) \tag{3.6d}$$

$$\mathcal{C}_n^{(L)} = \sum_r N_n^r \, \mathrm{tr}_{L_n^r}(T_n) \tag{3.6e}$$

$$\mathcal{C}_{na}^{(L)} = \sum_r N_n^r \, \mathrm{tr}_{L_n^r}(Q_n^a) \; . \tag{3.6f}$$

Here T_n denotes any generator in the $\mathcal{H}_n$ parts of the gauge group. Recall that Q_n^a are the generators of the $U_I(1)$ gauge groups. Furthermore, L_n^r denotes the possible matter representations of the low energy gauge group $\mathcal{G}_n$ on the orbifold plane n, that is, the representations appearing in the decomposition (2.27). Fortunately, all that enters the anomaly is the chiral asymmetry N_n^r of those representations which can be expressed in terms of the internal bundles via the index theorem (2.28). The first four anomaly coefficients above cover the possible combinations of $U_I(1)$ gauge fields with gauge fields in $\mathcal{H}_n$. The final two coefficients measure the mixed gravitational anomalies of $\mathcal{H}_n$ and the $U_I(1)$ fields, respectively. Note that although there are no purely gravitational anomalies in five dimensions, the mixed gravitational-gauge anomalies involve the orbifold planes, which are four-dimensional. In four dimensions, mixed gravitational–gauge anomalies can arise [48, 49], which is precisely what happens in (3.6e,3.6f).

Now, following Ref. [34, 35], we replace N_n^r in the above anomaly coefficients using the index theorem and further simplify the expressions by means of the trace formulae collected in Appendix A. This allows one to completely express the anomaly coefficient in terms of topological Calabi–Yau and bundle data. We find

$$\begin{aligned}
\mathcal{C}_n &= 0 &\quad (3.7a)\\
\mathcal{C}_{na} &= \mp\frac{1}{8\pi}\eta^i_{na}\beta_i &\quad (3.7b)\\
\mathcal{C}_{nab} &= 0 &\quad (3.7c)\\
\mathcal{C}_{nabc} &= \frac{3}{8\pi}\left[\mp\beta_i\eta^i_{n(a}\delta_{bc)}+\frac{1}{12\pi^2}d_{ijk}\eta^i_{na}\eta^j_{nb}\eta^k_{nc}\right] &\quad (3.7d)\\
\mathcal{C}^{(L)}_n &= 0 &\quad (3.7e)\\
\mathcal{C}^{(L)}_{na} &= \frac{3}{2\pi}\left(\mp\beta_i-\frac{1}{12}\gamma_i\right)\eta^i_{na}\ . &\quad (3.7f)
\end{aligned}$$

As before, the upper (lower) sign refers to the first (second) orbifold plane. The quantum variation due to the triangle diagrams is then given by

$$\delta_{\rm Q}S_5=\frac{1}{16\pi^2}\sum_{n=1}^{2}\int_{M_4^{(n)}}\Lambda^a_n\left[\mathcal{C}_{na}{\rm tr}F_n^2-\frac{1}{24}\mathcal{C}^{(L)}_{na}{\rm tr}R^2+\frac{1}{3}\mathcal{C}_{nabc}\mathcal{F}^b_n\mathcal{F}^c_n\right] \tag{3.8}$$

with the anomaly coefficients $\mathcal{C}$ as given in Eqs. (3.7a–3.7f). This quantum variation cancels the classical one in Eq. (3.5) except for the first term in the cubic anomaly coefficient. In the quantum variation (3.8), this first term is symmetrized in all three indices while this is not the case in the classical variation (3.5). This, however, is not a problem. The form of the anomaly in Eq. (3.8) that we have chosen corresponds to a specific way of regularizing the triangle diagrams. Using the known ambiguity in the four–dimensional anomaly, we can, however, regularize the diagrams associated to the cubic anomaly differently and put them exactly in the appropriate form so as to cancel the classical variation (3.5). Hence, we indeed have $(\delta_{\rm cl}+\delta_{\rm Q})S_5=0$ and the total anomaly cancels.

3.3 Discussion

Let us now discuss the significance of the above results in some detail. First, we would like to do this in relation to the 11–dimensional origin of the five–dimensional theory.

As we have seen, the anomalous classical variation originates from the first three terms in the topological part (2.31) of the five–dimensional bulk action. The 11–dimensional origin of these terms is as follows

$$\begin{aligned}
C\wedge X^8 &\rightarrow \gamma_i\,\mathcal{B}^i{\rm tr}R^2 &\quad (3.9)\\
CGG \text{ with non-zero mode} &\rightarrow \beta_i\,\mathcal{B}^i\wedge G &\quad (3.10)\\
CGG &\rightarrow d_{ijk}\mathcal{B}^i\mathcal{D}^j\mathcal{D}^k\ . &\quad (3.11)
\end{aligned}$$

Recall here that CGG is the Chern–Simons term of 11–dimensional supergravity (2.3) while $C\wedge X_8$ is the Green–Schwarz term defined in Eq. (2.5). The non–zero mode that, inserted into CGG, leads to the second term above, is a purely internal configuration for the 11–dimensional antisymmetric tensor fields strength G. It was explicitly given in Eq. (2.21). The crucial ingredient forcing us to assign gauge transformations to the bulk antisymmetric tensor fields $\mathcal{B}^i$ was the Bianchi identity (2.39). It directly results from the 11–dimensional Bianchi identity (2.7) with two of the indices being taken internal and the others external. As we have seen, given such a configuration of indices, the sources in the five–dimensional Bianchi identities for $\mathcal{D}^i$ can only receive non–trivial contributions from the $U_I(1)$ gauge fields. As a consequence, only those $U_I(1)$ fields appear on the right hand side of (2.39) and the bulk vector fields $\mathcal{B}^i$ transform under the type I gauge transformation only. What is the individual contribution of the above three bulk terms to the anomalous variation? The first term (3.10), originating from the 11–dimensional Green–Schwarz term, causes only the second contribution proportional to γ_i in the mixed gravitational anomaly in Eq. (3.5). Likewise, the third term (3.11), originating from the 11–dimensional Chern–Simons term, leads to only one term, namely the second one in the cubic anomaly in Eq. (3.5). The most important part of the anomalous variation results from the second term (3.10). It leads to all terms in Eq. (3.5) proportional to β_i and, hence, contributes to all three types of anomalies that we have encountered. This shows again the importance of this term, which is also responsible for the gauging of the supergravity theory [28, 30]. Had we missed this term, or equivalently, had we missed the underlying non–zero mode, we would clearly be unable to uncover the five–dimensional anomaly structure.

Let us now move on to discuss this five–dimensional structure in detail. To do this, let us focus on the expression (3.8) for the quantum variation. The non–trivial information resides in the comparison between the general definition (3.6a–3.6f) of the anomaly coefficients and the expressions (3.7a–3.7f) that one obtains for those coefficients within the context of heterotic M–theory. A priori, from their definition (3.6a–3.6f), the anomaly coefficients could have taken any value depending on the gauge groups and particle contents on the orbifold planes. We are told, however, that in heterotic M–theory they have a precise structure in terms of the underlying compactification. More concretely, they are given by the Calabi–Yau intersection numbers d_{ijk}, the topological numbers β_i and γ_i related to second Chern characters of the internal gauge and tangent bundles and the coefficients η^i_{na} that specify the $U(1)$ parts of the internal gauge bundles. Clearly, this structure reflects the restrictions that heterotic M–theory imposes on the particle content residing on the orbifold planes. In a “bottom up approach” we can also turn this around and use the anomaly structure to learn about these restrictions. For a more precise discussion, we first remark that the expressions (3.7a–3.7f) are rather symmetric with respect to the two orbifold planes. It is worth pointing out that the two gauge theories on those planes are completely hidden with respect to one another, that is, they only interact gravitationally. Correspondingly, anomaly cancellation works completely independently for the two planes. Obviously, there cannot be any $U(1)$ gauge fields "extending over both planes" not to mention $U(1)$ gauge fields that "got displaced into the bulk in between". Consequently, we have only non–vanishing triangle diagrams that couple gauge currents on the *same* plane. Our subsequent discussion, focusing on a single orbifold plane, therefore applies to either one of the two orbifold planes.

The first crucial observation is that some of the anomaly coefficients vanish within heterotic M–theory. This implies that the classical as well as the quantum variation should be zero individually. Specifically, we see from Eq. (3.7a) and (3.7e) that the diagrams coupling three currents in the $\mathcal{H}_n$ part of the gauge group as well as the mixed gravitational anomaly with one $\mathcal{H}_n$ and two gravity currents vanish. In this sense, the $\mathcal{H}_n$ part of the gauge group is anomaly–free. Recall that $\mathcal{H}_n$ contains the complete non–abelian part of the gauge group as well as the $U_{II}(1)$ fields of type II. It is well–known that the non–abelian part of the gauge group is anomaly–free [34]. In addition, we learn here that the same is true for $U_{II}(1)$ factors of type II. As a consequence, non–vanishing diagrams always involve the $U_I(1)$ fields of type I. Among those diagrams the one coupling two $U_I(1)$ fields to a field in $\mathcal{H}_n$ vanishes from Eq. (3.7c). This is trivial for the non–abelian part of $\mathcal{H}_n$ (since then $\mathrm{tr}_{L_n^r}(T_n) = 0$ always). However, it gives a non–trivial information for the $U_{II}(1)$ factors in $\mathcal{H}_n$, namely that the $U_I(1)^2 U_{II}(1)$ triangle diagram vanish. We are left with three types of anomaly that can be non–zero in general. These are the mixed $U_I(1)$ gauge anomaly (3.7b), the $U_I(1)^3$ cubic anomaly (3.7d) and the mixed $U_I(1)$ gravitational anomaly (3.7f). From the general form of the mixed $U_I(1)$ gauge anomaly, there could in principle be a dependence on the specific generator T_n of $\mathcal{H}_n$ that has been chosen. However, Eq. (3.7b) shows that the anomaly coefficient is, in fact, independent of this choice. Hence, we conclude that the mixed $U_I(1)$ gauge anomaly coefficient is the same for all non–abelian factors and all $U_{II}(1)$ factors in $\mathcal{H}_n$. In this sense, the mixed gauge anomaly may be called universal.

How many anomalous $U_I(1)$ symmetries are there? As we have seen, we start with a given number k_n of $U_I(1)$ factors on each orbifold plane $n = 1, 2$ which is determined by the specific compactification. Not all of them, however, necessarily have to be anomalous. It is clear from Eq. (3.7b), for example, that we can always choose a basis such that only one of the $U_I(1)$ factors has a mixed gauge anomaly while this type of anomaly vanishes for all the other $U_I(1)$. Here we stress again that this statement as well as the subsequent ones apply to each orbifold plane separately. Hence, in sum, we may have two $U_I(1)$ groups with mixed gauge anomalies, one on each orbifold plane. Each of these $U_I(1)$'s may have mixed gravitational and cubic anomalies at the same time. Those are fixed in terms of topological data by Eq. (3.7d) and (3.7f). However, these expressions also show that the three anomaly coefficients are not related in any universal way. That is to say, without further information about the type of compactification and the associated topological data, they are practically independent. We have therefore identified one $U_I(1)$ factor on each orbifold plane that has a universal mixed gauge anomaly (independent of the gauge factor) as well as generally unrelated cubic and mixed gravitational anomalies. While the remaining $k_n - 1$ $U_I(1)$ factors are now free of mixed gauge anomalies by construction, they may have gravitational and cubic anomalies, however. Again, we can diagonalize these remaining $U_I(1)$'s such that at most one of them has a mixed gravitational and a cubic anomaly. Finally, then, the other $k_n - 2$ $U_I(1)$'s which are now free of mixed gauge and gravitational anomalies may have cubic anomalies only.

Let us summarize this discussion briefly. Recall that $\mathcal{H}_n$ consist of the non–abelian and the $U_{II}(1)$ gauge fields, collectively called A_n. Furthermore, we denote $U_I(1)$ gauge fields by $\mathcal{A}_n^a$, where $a = 1, \ldots, k_n$ and the graviton by g. We then refer to a triangle diagram by specifying the triple of gauge fields to which it couples. Then the anomaly structure on *each* orbifold plane in five–dimensional heterotic M–theory is as follows :

- The $\mathcal{H}_n$ part of the gauge group is anomaly–free in the sense that the A_n^3 and $A_n gg$ anomalies vanish.
- The mixed $\mathcal{A}_n^a \mathcal{A}_n^b A_n$ anomalies vanish.
- After a suitable choice of basis there is at most a single $U_I(1)$ gauge field, say $\mathcal{A}_n^1$ with all three remaining types of anomalies non–vanishing. That is, we may have anomalies of type $\mathcal{A}_n^1 A_n A_n$, $\mathcal{A}_n^1 \mathcal{A}_n^a \mathcal{A}_n^b$ and $\mathcal{A}_n^1 gg$. In terms of the topological data the associated anomaly coefficients are given in Eq. (3.7b), (3.7d) and (3.7f). The remaining $U_I(1)$ factors are free of mixed gauge anomalies, that is the $\mathcal{A}_n^a A_n A_n$ anomaly vanishes for $a > 1$.
- The coefficient of $\mathcal{A}_n^1 A_n A_n$ is independent of which specific gauge field A_n within $\mathcal{H}_n$ is considered. Apart from this restriction, the three non–vanishing anomaly coefficients are generically unrelated.
- After another choice of basis there is at most one among the remaining $U_I(1)$ symmetries, say $\mathcal{A}_n^2$, with mixed gravitational and cubic anomaly. In other words, the anomalies of type $\mathcal{A}_n^2 gg$ and $\mathcal{A}_n^2 \mathcal{A}_n^a \mathcal{A}_n^b$ can be non–vanishing. Again the two associated anomaly coefficients are generically unrelated. All other $\mathcal{A}_n^a$ for $a > 2$ have cubic anomalies at most.

4 Five–dimensional anomaly cancellation – a different viewpoint

In the previous section, we have analyzed anomaly constraints on the particle spectrum residing on the orbifold planes of five–dimensional heterotic M–theory. In doing so we have used all of the knowledge arising from the 11–dimensional Hořava–Witten construction upon descent down to five dimensions. The derivation of this 11–dimensional theory and, in particular, the E_8 gauge multiplets to which it couples is, however, based upon anomaly cancellation as well. A natural question, therefore, is whether one can avoid using some of this 11–dimensional knowledge and derive the anomaly constraints directly on a five–dimensional basis. Concretely, we would like to start from pure 11–dimensional supergravity, that is, we are not coupling any E_8 gauge fields to the theory. We can then consider this theory in the background of a Calabi–Yau three–fold and a non–zero mode configuration of the antisymmetric tensor field. The resulting effective action is a five–dimensional $\mathcal{N} = 1$ gauged supergravity, given by the bulk part of the action that we have described in the previous section. Subsequently, we would like to consider this five–dimensional theory on the orbifold S^1/Z_2. Our main question is then the following. Which constraints does five–dimensional anomaly cancellation impose on the "twisted states" residing on the two four–dimensional fixed planes of this orbifold? Hence, we are asking, in analogy with the original 11–dimensional construction, how five–dimensional M–theory can be put on an orbifold consistently. Given that one is usually bound to use anomaly cancellation as the basic tool, this question might be no less fundamental than the corresponding one in 11 dimensions.

4.1 Starting point – five–dimensional gauged supergravity

We start with the action of 11–dimensional supergravity (2.3) including the Green–Schwarz term (2.5). We would like to consider this theory on a space–time background of the structure

$$M_{11} = X \times S^1 \times M_4 \tag{4.1}$$

where X is a Calabi–Yau three–fold and M_4 is four–dimensional Minkowski space. In addition to the metric background, we would also like to allow a non–zero mode background for the antisymmetric tensor field strength G in the internal Calabi–Yau space. Given that G has to be an element of $H^4(X)$ the most general form of such a background is given by

$$\bar{G} = 4\pi^2\lambda\,\beta_i\nu^i\,\epsilon(y)\ . \tag{4.2}$$

Recall here that $\{\nu^i\}_{i=1,\ldots,h^{1,1}}$ is a basis of $H^4(X)$, the constant λ was defined in Eq. (2.1) and $\epsilon(y)$ is the step-function. The coordinate of the circle S^1 is called y and we use the same conventions as previously for the orbifold coordinate.

As we have mentioned previously in section 2, the appearance of the step function $\epsilon(y)$ in the background (4.2) is motivated by the wish to leave the $y \to -y$ Z_2 symmetry unbroken. Making a generalized Kaluza-Klein ansatz such as (4.2) without the $\epsilon(y)$ would break the Z_2 symmetry because G is Z_2 odd. This choice is also motivated by the wish to have interesting solutions to the resulting $D = 5$ field equations, because these field equations acquire a cosmological potential term [28, 30] which rules out flat space, or indeed any maximally symmetric space, as a solution. Including the $\epsilon(y)$ in the ansatz (4.2) allows solutions that have the character of 3-brane domain walls from a $D = 5$ perspective, although from a $D = 11$ perspective they appear as 5-branes wrapped around two-cycles of the Calabi–Yau space and piled into stacks for reduction in the other four Calabi–Yau directions [28, 30]. This purely field-theoretic background solution accords precisely with the Hořava–Witten orbifold structure that appears to be a non-intrinsic injection into the theory from the $D = 11/D = 10$ perspective of section 2. Thus, in $D = 5$, the orbifold becomes a natural aspect of the field-theoretic background.

The form (4.2) for the non–zero mode is identical to the one we used in section 2, Eq. (2.21). An important difference, however, is that the constants β_i, although quantized as usual, are here taken to be otherwise undetermined. In section 2, the non–zero mode was forced upon us due to gauge field and gravity sources in the non–trivial Bianchi identity of Hořava–Witten construction. Correspondingly, the β_i were determined in terms of those sources by Eq. (2.13). Here, the non–zero mode and the coefficients β_i in particular may be freely chosen. For example, they could be set to zero if desired.

The form (4.2) of the Kaluza-Klein ansatz corresponds to having just two regions in the y coordinate, with equal and opposite non-zero mode charges. This choice can be extended by the inclusion of additional extended objects at various y values, by the inclusion of additional step functions in ansätze such as (4.2). All such cases will be constrained, however, by a cohomological condition [33]

$$\sum_{\mathcal{S}^1 \text{ patches } (n)} \beta_i^{(n)} = 0 \ . \tag{4.3}$$

This is derived by requiring that, although the Bianchi identity for G is modified by the inclusion of delta–function terms such as in (2.7), the form dG should still be exact in the full $D = 11$ spacetime, hence requiring that its integral vanish when integrated over any closed cycle. The ansatz (4.2) gives the simplest nontrivial solution to this cohomological requirement.

The five–dimensional effective theory on $M_5 = S^1 \times M_4$ obtained by reducing on the Calabi–Yau space with the generalized ansatz (4.2) is given by gauged $\mathcal{N} = 1$ supergravity. We are interested here in the parts of this action involving the antisymmetric tensor field. These are identical to the bulk parts of the action (2.29) used in section 2 which we repeat here for convenience. The kinetic and topological terms are given by

$$2\kappa_5^2\, S_{\text{kin}} = -\int_{M_5} \left[2G_{ij}\mathcal{D}^i \wedge *\mathcal{D}^j + 2V^{-1}X \wedge *X^* + V^2 G \wedge *G\right] \tag{4.4}$$

and

$$\begin{aligned}2\kappa_5^2\, S_{\text{top}} = -\sqrt{2}\int_{M_5} \Big[&\frac{\pi^2\lambda^2}{6v^{2/3}}\, \gamma_i\, \mathcal{B}^i \wedge \text{tr}R^2 - \frac{8\pi^2\lambda}{v^{2/3}}\epsilon(y)\, \beta_i\, \mathcal{B}^i \wedge G + \frac{1}{3}d_{ijk}\mathcal{B}^i\mathcal{D}^j\mathcal{D}^k \\ &+ i(\xi G \wedge X^* - \xi^* G \wedge X)\Big] \ . \end{aligned} \tag{4.5}$$

We note that the constant γ_i are proportional to the second Chern classes of the Calabi–Yau tangent bundle and have been defined in Eq. (2.14). The constants d_{ijk} are the Calabi–Yau intersection numbers (2.20). Hence, both γ_i and d_{ijk} are still given to us by topological properties of the compactification. The non-zero mode charges β_i are topological quantities related to the quantized flux of antisymmetric tensor fields on the internal manifold, but are not related to the topology of the Calabi-Yau space itself. Another difference with respect to the previous section concerns the viewpoint one takes with respect to the modified Bianchi identities. Here, this modification is seen to follow from the ansatz (4.2), corresponding to the inclusion of magnetically charged objects located at the step–function jump points $y = 0$, $y = \pm\pi\rho$. In section 2, the modifications to the Bianchi identity were prescribed externally as a consequence of the Hořava–Witten $D = 11/D = 10$ orbifold construction. Note that the modification to the Bianchi identity that we have so far, just from the background ansatz (4.2), is still quite modest, corresponding just to the usual source for a static magnetically charged extended object without further worldvolume excitations. Moreover, the Bianchi identity modifications implied by (4.2) have non–vanishing projections only with four indices in the Calabi-Yau directions, and the remaining one in the orbifold y direction. The Bianchi identity modifications presented in section 2, on the other hand, are more extensive, containing also sources for orbifold-plane modes that we have not yet seen from the $D = 5$ field-theoretic viewpoint, and which have projections with all indices in the $D = 5$ space. To see the rôle of such modes, we now turn to a closer inspection of the modified Bianchi identities.

4.2 Orbifolding and modifying the Bianchi identities

We would now like to consider solutions to the above $D = 5$ theory having the space–time character

$$M_5 = S^1/Z_2 \times M_4 \tag{4.6}$$

that is including a circle orbifolded by Z_2. Our conventions here are the same as those described in the beginning of section 2.1. We shall accordingly have two four–dimensional orbifold planes $M_4^{(n)}$, where $n = 1, 2$, located at the step–function jump points $y = y_1 \equiv 0$ and $y = y_2 \equiv \pi\rho \sim -\pi\rho$. It is then easy to see from (4.5) that $\mathcal{B}^i$ and G have to be Z_2 odd fields [3]. We already know that there exists a static BPS double domain wall solution of the bulk supergravity with the three–brane domain walls identified with the two orbifold planes [28, 30].

Let us now assume the existence of "twisted" states on these orbifold planes. The five–dimensional bulk gravitino, corresponding to eight states, is subject to a chirality constraint on the orbifold planes. Hence, on these planes we have four–dimensional $\mathcal{N} = 1$ supersymmetry corresponding to four supercharges. We start by assuming a set of $\mathcal{N} = 1$ gauge fields on each plane with associated gauge groups

$$\mathcal{G}_n = \mathcal{H}_n \times \mathcal{J}_n \,. \tag{4.7}$$

[3] We call an antisymmetric tensor field even (odd) if its components orthogonal to the orbifold are even (odd).

Here $\mathcal{H}_n$ is the semi–simple part and the $U(1)$ factors are collected in $\mathcal{J}_n$. We denote $\mathcal{H}_n$ gauge fields by A_n^α with field strength F_n^α where α runs over the simple factors. Furthermore the $U(1)$ fields are denoted by $\mathcal{A}_n^a$ with field strengths $\mathcal{F}_n^a$ where a labels the various $U(1)$ factors. Also we assume the existence of $\mathcal{N}=1$ chiral multiplets on each plane transforming in the representations L_n^r of $\mathcal{G}_n$. Of course the configuration we choose cannot be completely arbitrary since the full theory has to be anomaly–free. More precisely, anomalies should cancel on each plane individually. While this can be done, of course, by choosing a field content with vanishing triangle anomalies on each plane we know from the previous experience that this is not the most general case. We may allow for some non–vanishing triangle anomalies provided they can be cancelled by anomaly inflow from the bulk. We know that such an inflow should be generated by a non–trivial variation of the topological part (4.5) of the action due to a gauge transformation of the antisymmetric tensor fields. Such a gauge transformation may originate from a non–trivial Bianchi identity with sources supported on the orbifold planes and depending on orbifold gauge fields. To incorporate this possibility, we would like to modify the Bianchi identity in the most general way. We write

$$dG = -\lambda \sum_{n=1}^{2} J_n \wedge dy\, \delta(y-y_n) \tag{4.8}$$

$$d\mathcal{D}^i = -2v^{-1/3}\lambda \sum_{n=1}^{2} I_n^i \wedge dy\, \delta(y-y_n) \tag{4.9}$$

with four–forms J_n and two–forms I_n^i which reside on the orbifold plane. They should be given in terms of orbifold gauge fields eventually but are kept arbitrary for the moment. Note that we have not considered the Bianchi identity for the field ξ here. The reason is that the smallest gauge–invariant form on the orbifold planes is a two–form which does not fit on the right hand side of the ξ Bianchi identity. We also write the integrated form of the above identities

$$G = dC - \lambda \sum_{n=1}^{2} w_n \wedge dy\delta(y-y_n) \tag{4.10}$$

$$\mathcal{D}^i = d\mathcal{B}^i - 2v^{-1/3}\lambda \sum_{n=1}^{2} v_n^i \wedge dy\delta(y-y_n) \tag{4.11}$$

where the "Chern–Simons forms" w_n and v_n^i satisfy

$$dw_n = J_n\ , \qquad dv_n^i = I_n^i\ . \tag{4.12}$$

4.3 Variation of the action

To compute the classical gauge variation of the action, we first note that the topological part of the action (4.5) depends on C only through its field strength G. As in the previous section, we are therefore only concerned with the transformation of $\mathcal{B}^i$. We start with some transformation law

$$\delta v_n^i = d\lambda_n^i \tag{4.13}$$

for the forms v_n^i, where the λ_n^i are transformation parameters to be determined later. Requiring $\delta\mathcal{D}^i = 0$, we find from Eq. (4.11) that

$$\delta\mathcal{B}^i = 2v^{-1/3}\lambda\sum_{n=1}^{2}\lambda_n^i dy\,\delta(y-y_n)\;. \tag{4.14}$$

To compute the variation of the action we will also need the behavior of the field strengths close to the orbifold planes. From Eqs. (4.8) and (4.9) and the associated equations of motion one finds

$$\begin{aligned} dy\wedge G\,|_{y=y_n} &= \mp\frac{\lambda}{2}J_n\wedge dy & (4.15)\\ dy\wedge\mathcal{D}^i\,|_{y=y_n} &= \mp\lambda v^{-1/3}I_n^i\wedge dy\;. & (4.16)\end{aligned}$$

The classical variation of the bulk action is then given by

$$\delta_{\rm cl}S_5 = -\frac{c^3}{128\pi^3}\sum_{n=1}^{2}\int_{M_4^{(n)}}\lambda_n^i\left[\frac{1}{12}\gamma_i{\rm tr}R^2\mp 2\beta_i J_n + d_{ijk}I_n^j\wedge I_n^k\right]\;. \tag{4.17}$$

To proceed further, we obviously need to know more about the gauge field parts of the sources J_n and I_n^i. Its most general form is given by an arbitrary linear combination of all gauge invariant forms of the correct degree. This leads to

$$\begin{aligned} J_n &= \sum_\alpha k_{n\alpha}{\rm tr}(F_n^\alpha)^2 + \sum_{a,b}h_{nab}\mathcal{F}_n^a\mathcal{F}_n^b - l_n{\rm tr}R^2 & (4.18)\\ I_n^i &= \eta_{na}^i\mathcal{F}_n^a & (4.19)\end{aligned}$$

where $k_{n\alpha}$, h_{nab}, l_n and η_{na}^i are arbitrary constants. Here, the index α runs over the various simple groups contained in the semi–simple part $\mathcal{H}_n$ of the gauge group. Writing the transformation parameters λ_n^i as

$$\lambda_n^i = \eta_{na}^i\Lambda_n^a \tag{4.20}$$

with new parameters Λ_n^a we learn from Eq. (4.19) that our initial transformation (4.13) actually describes the $U(1)$ gauge variation

$$\delta\mathcal{A}_n^a = d\Lambda_n^a\;. \tag{4.21}$$

Hence, similarly as in the previous section, the bulk vector fields $\mathcal{B}^i$ transform under the $U(1)$ gauge symmetries only. This also implies that triangle anomalies not involving those $U(1)$ field must be zero as they cannot be cancelled by anomaly inflow. In this sense, the semi–simple parts $\mathcal{H}_n$ of the gauge groups must be anomaly–free on both orbifold planes. Depending on the coefficients η^i_{na} in Eq. (4.19) some of the $U(1)$ gauge fields might not enter the Bianchi identity (4.9). To bring our conventions in line with the previous section, we will call such $U(1)$ fields to be of type II and add them to the semi–simple parts $\mathcal{H}_n$ of the gauge group. All $U(1)$ fields that do contribute to the Bianchi identity will be called type I and remain in the $\mathcal{J}_n$ part.

Then, inserting the above Ansätze into the variation (4.17) we arrive at

$$\delta_{\rm cl} S_5 = -\frac{c^3}{128\pi^3}\sum_{n=1}^{2}\int_{M_4^{(n)}}\Lambda^a_n\Bigg[(\mp\beta_i\eta^i_{na})\sum_\alpha k_{n\alpha}{\rm tr}(F^b_n)^2 - \left(\mp 2l_n\beta_i - \frac{1}{12}\gamma_i\right)\eta^i_{na}{\rm tr}R^2 + \left(\mp 2\beta_i\eta^i_{na}h^{bc}_n + \frac{1}{6\pi^2}d_{ijk}\eta^i_{na}\eta^j_{nb}\eta^k_{nc}\right)\mathcal{F}^b_n\mathcal{F}^c_n\Bigg] \qquad (4.22)$$

for the classical variation of the action. The meaning of the three terms above is obvious. They correspond to a mixed $U_I(1)$ gauge anomaly, a mixed $U_I(1)$ gravitational anomaly and a $U_I(1)$ cubic anomaly. This expression has to be compared with the quantum variation due to triangle diagrams on each boundary. Let us define the triangle anomaly coefficient $\mathcal{C}$ as in Eqs. (3.6a–3.6f). Unlike in the previous section, we have no further information about the particle content on the orbifold plane. We simply require, at this point, the associated triangle anomaly to be such that it is cancelled by the anomaly inflow (4.22). This gives the following specific form for the anomaly coefficients

$$\mathcal{C}_n = 0 \qquad (4.23a)$$

$$\mathcal{C}_{na,\alpha} = \mp\frac{1}{8\pi}\eta^i_{na}\beta_i k_{n\alpha} \qquad (4.23b)$$

$$\mathcal{C}_{nab} = 0 \qquad (4.23c)$$

$$\mathcal{C}_{nabc} = \frac{3}{8\pi}\left[\mp\beta_i\eta^i_{n(a}h_{nbc)} + \frac{1}{12\pi^2}d_{ijk}\eta^i_{na}\eta^j_{nb}\eta^k_{nc}\right] \qquad (4.23d)$$

$$\mathcal{C}^{(L)}_n = 0 \qquad (4.23e)$$

$$\mathcal{C}^{(L)}_{na} = \frac{3}{2\pi}\left(\mp 2l_n\beta_i - \frac{1}{12}\gamma_i\right)\eta^i_{na}\,. \qquad (4.23f)$$

The associated quantum variation of the gauge theories on the orbifold planes then takes the form

$$\delta_{\rm Q} S_{\rm bound} = \frac{1}{16\pi^2}\sum_{n=1}^{2}\int_{M_4^{(n)}}\Lambda^a_n\left[\sum_\alpha C_{na,\alpha}{\rm tr}(F^\alpha_n)^2 - \frac{1}{24}C^{(L)}_{na}{\rm tr}R^2 + \frac{1}{3}C_{nabc}\mathcal{F}^b_n\mathcal{F}^c_n\right]\,. \qquad (4.24)$$

4.4 Discussion

Let us now discuss our results. We would like to compare the general structure of anomalies, as given in Eqs. (3.6a–3.6f), with the specific form (4.23a–4.24) that we have found by considering five–dimensional gauged supergravity (as obtained from 11–dimensional supergravity) on an orbifold. Also, we would like to compare those constraints to the corresponding ones (3.7a–3.7f) that we have obtained within heterotic M–theory. As far as the relation of the two orbifold planes is concerned, the general discussion in subsection 3.3 applies. Anomaly cancellation works independently on both planes and the subsequent discussion should be applied to either one of them.

Let us first discuss the vanishing anomaly coefficients in our list. From Eq. (4.23a–4.23f) those are the cubic anomalies for the $\mathcal{H}_n$ part of the gauge group, the mixed gravitational anomaly of $\mathcal{H}_n$ and the mixed anomaly of one $\mathcal{H}_n$ field with two $U(1)$ fields. In particular, we conclude that the semi–simple part of the orbifold gauge groups always has to be anomaly–free. This structure of vanishing coefficients is the same as in heterotic M–theory as given in Eqs. (3.7a–3.7f). In both cases we remain with three types of non–vanishing coefficient. While the structure of those remaining coefficients is similar, there are also some significant differences. To discuss these it is useful to distinguish various classes of parameters that determine those coefficients. First, there is topological data of the Calabi–Yau space, namely the second Chern class of the Calabi–Yau tangent bundle γ_i and the intersection numbers d_{ijk}. Clearly this data is determined in terms of the underlying compactification in both cases and it enters the respective formulae in a very similar way. Secondly, there are the coefficients β_i and η^i_{na}. They appear in a very similar way in the anomaly coefficients, but their interpretation is different in the two cases. While they are determined in terms of topological data of the internal vector bundles in the case of heterotic M–theory, they are merely just parameters here. While the β_i parameterize the non–zero mode that we have put in to compactify 11–dimensional supergravity, the η^i_{na} appear as free parameters in our Ansatz (4.19) for the sources in the Bianchi identities. Thirdly, we have the parameters $k_{n\alpha}$, h_{nab} and l_n that appear in the Ansatz (4.18) for the sources. Those parameters arise because of our lack of knowledge about the precise form of the Bianchi identity for the four–form G in the present case. In heterotic M–theory they were determined exactly, as comparison of the Ansatz (4.18) with Eq. (2.36) shows.

As a result, we have a number of significant differences with respect to the earlier discussions. Some "numerical freedom" in the anomaly coefficient as compared to heterotic M–theory is introduced by the arbitrariness of the second group of parameters, that is, β_i and η^i_{na}. However, given that one cannot completely classify the allowed values for those parameters even for heterotic M–theory (as they depend, for example, on the choice for the Calabi–Yau manifold) it is hard to quantify this difference. More significant are some of the differences related to parameters in the third class. We note, however, that h_{nab} can be absorbed into a redefinition of the $U(1)$ fields $\mathcal{A}^a_n$. While the parameters l_n make the mixed gravitational anomaly (4.23f) more flexible, the crucial difference comes from the parameters $k_{n\alpha}$ in Eq. (4.23b). Recall, here, that the index α labels the various simple groups in $\mathcal{H}_n$ and, according to our above convention, also the various $U_{II}(1)$ factors of type II in $\mathcal{H}_n$. The presence of the parameters $k_{n\alpha}$ indicates that the mixed $U_I(1)$ gauge anomaly (4.23b) can be different for each of those factors. Such a non–universal mixed anomaly is rather different from what we found in heterotic M–theory (compare with Eq. (3.7b) where it was always gauge–factor independent. Finally, a crucial difference concerns the size of the orbifold gauge groups. While, within the context of heterotic M–theory, they are bound to fit within E_8 on each plane no such restriction arises in the present context.

To summarize, we have found anomaly cancellation in the setting of this section to be more flexible than in five–dimensional heterotic M–theory in two important ways. Firstly, non–universal mixed $U(1)$ gauge anomalies are possible. Secondly, the orbifold groups are not restricted to fit into E_8. It is clear that both generalizations can be of considerable phenomenological importance. An important question is whether such generalizations have an interpretation in terms of M–theory. It would be very interesting to search for such an interpretation, for example in the context of heterotic M–theory models with five–branes [33, 36, 37, 38]. However, we will not pursue this further in the present paper.

5 Heterotic anomaly cancellation in four dimensions

In section 3 we have analyzed $E_8 \times E_8$ heterotic anomaly cancellation from a five–dimensional viewpoint. However, upon dimensional reduction five–dimensional heterotic M–theory reproduces the effective four–dimensional action of the $E_8 \times E_8$ heterotic string [30], at least to one–loop order. Hence, the results for the structure of anomaly cancellation that we found in five dimensions should be directly applicable to the four–dimensional theory. This seems rather puzzling, however, since this structure contradicts in various ways what is commonly assumed about anomaly cancellation in the four–dimensional $E_8 \times E_8$ heterotic string. Usually, it is stated that there is at most one anomalous $U(1)$ symmetry that "extends" over the hidden and the observable sector. Its quantum anomaly is supposed to be cancelled exclusively due to a non–trivial transformation law of the dilaton superfield. As a consequence of the universal dilaton coupling, in order for such a cancellation to work, all triangle anomaly coefficients (including the cubic and the mixed gravitational one) have to be in fixed proportions. On the other hand, the structure that we found in five dimensions turned out to be significantly more flexible. Various anomalous $U(1)$ symmetries were possible, hidden and observable sectors were greatly independent, and some of the triangle anomaly coefficients were generically unrelated. Moreover, the cancellation mechanism was due to a transformation of the five–dimensional vector fields, which, from a four–dimensional viewpoint, are associated with the T moduli rather than with the dilaton. As we will see, these discrepancies are resolved by a careful distinction between $U(1)$ symmetries of type I and type II and their properties. Firstly, however, we would like to study four–dimensional anomaly cancellation systematically in order to confirm our previous results and gain some confidence. The effective four–dimensional action that we are going to need for this can be obtained in two ways. Firstly one can reduce the 10–dimensional

$E_8 \times E_8$ heterotic action on a Calabi–Yau three–fold to four dimensions. Secondly, we can start with the five–dimensional heterotic M–theory as given in section 2 and reduce it to four dimensions. Both methods have to agree since the five– and the four–dimensional theories, as well as the 11– and 10– dimensional theories are equivalent. This has been explicitly verified in references [30] and [46], respectively. Here we will use the first approach, if only to have a better comparison to the weakly coupled heterotic string.

5.1 The 10–dimensional action

We are interested in the part of the 10–dimensional $E_8 \times E_8$ heterotic effective action that involve the two–index antisymmetric tensor field B with field strength $H = dB + \cdots$ as well as certain terms involving the curvature and the $E_8 \times E_8$ gauge fields A with fields strength F. This part of the action is given by

$$S_{10} = S_{\rm kin} + S_{\rm top} \tag{5.1}$$

where

$$\begin{aligned} S_{\rm kin} &= -\frac{1}{2\kappa_{10}^2}\int_{M_{10}} \left[e^{-2\phi} H \wedge *H + \frac{\alpha'}{4}({\rm tr}F^2 - {\rm tr}R^2)\right] \qquad (5.2)\\ S_{\rm top} &= -k \int_{M_{10}} B \wedge W_8 \ . \qquad (5.3)\end{aligned}$$

Here ϕ is the dilaton and the constant k is given by

$$k = \frac{c^3}{3\sqrt{2}\cdot 2^7 \pi^5 \alpha'} \ . \tag{5.4}$$

The non–trivial Bianchi identity for H reads

$$dH = -\frac{\alpha'}{2\sqrt{2}}({\rm tr}F^2 - {\rm tr}R^2) \ . \tag{5.5}$$

Furthermore, the anomaly polynomial W_8 has the well–known form

$$W_8 = \frac{1}{24}{\rm Tr}F^4 - \frac{1}{7200}\left({\rm Tr}F^2\right)^2 - \frac{1}{240}{\rm Tr}F^2{\rm tr}R^2 + \frac{1}{8}{\rm tr}R^4 + \frac{1}{32}\left({\rm tr}R^2\right)^2 \ . \tag{5.6}$$

As usual, Tr denotes the trace in the adjoint, while tr $=$ Tr/30. It is useful for our purpose to express the anomaly polynomial in terms of the individual E_8 gauge fields A_n with field strengths F_n. Writing $F = F_1 + F_2$, one obtains

$$W_8 = \frac{1}{4}\left({\rm tr}F_1^2\right)^2 + \frac{1}{4}\left({\rm tr}F_2^2\right)^2 - \frac{1}{4}{\rm tr}F_1^2{\rm tr}F_2^2 - \frac{1}{8}\left({\rm tr}F_1^2 + {\rm tr}F_2^2\right){\rm tr}R^2 + \frac{1}{8}{\rm tr}R^4 + \frac{1}{32}\left({\rm tr}R^2\right)^2 \ . \tag{5.7}$$

We would like to reduce the above theory on a background space–time

$$M_{10} = X \times M_4 \tag{5.8}$$

where X is a Calabi–Yau three–fold with metric[4] $\bar{g}_{10}$ and curvature two–form $\bar{R}$. As far as the gauge fields are concerned, the background configuration is exactly as described for the reduction of Hořava–Witten theory. We will not repeat this here but simply refer back to section 2.2.

Let us list the relevant zero modes around this background. We define the breathing modulus V to be

$$V = \frac{e^{-2\phi}}{v} \int_X \sqrt{\bar{g}_{10}} \; . \tag{5.9}$$

The $(1,1)$ moduli a^i_{10} appear in the expansion of the Kähler form $\Omega_{10} = a^i_{10}\Omega_i$ in terms of the basis $\{\Omega_i\}_{i=1,\cdots,h^{1,1}}$ of harmonic $(1,1)$ forms. For the antisymmetric tensor field and its field strength we write

$$B = \tilde{B} + \chi^i \Omega_i \tag{5.10}$$

$$H = \tilde{H} + X^i \wedge \Omega_i \tag{5.11}$$

where $\tilde{B}$ is the four–dimensional two–form with field strength $\tilde{H}$ and χ^i, where $i = 1, \cdots, h^{1,1}$ are axionic scalars with field strengths X^i. We have dropped contributions from $(2,1)$ modes which are not important in our context. In terms of four–dimensional multiplets, V and the dual of $\tilde{B}$ form the bosonic part of the dilaton superfield S, whereas a^i_{10} and χ^i represent the bosonic field content of the T^i moduli. For the gauge field zero modes originating from $E_8 \times E_8$, exactly the same discussion as in the 11–dimensional case applies. We adopt the notation of section 2.3 to which we refer back for details.

5.2 The four–dimensional effective action

By straightforward calculation, using the above setup, we find the part of the four–dimensional effective action relevant for anomaly cancellation

$$S_4 = S_{4,\rm kin} + S_{4,\rm top} \tag{5.12}$$

where the kinetic part is given by

$$S_{4,\rm kin} = -\frac{1}{2\kappa_4^2}\int_{M_4}\left[V^2 H\wedge *H + 2G_{ij}X^i\wedge *X^j\right] - \frac{1}{4g_0^2}\int_{M_4}\sqrt{-g}V\left[{\rm tr}F_1^2 + {\rm tr}F_2^2 + \sum_A (\mathcal{F}^A)^2\right] \; . \tag{5.13}$$

The topological part reads

$$S_{4,\rm top} = -6\pi^2 k\int_{M_4}\left[2\beta_i\hat{\eta}^i_A B\wedge\mathcal{F}^A + \chi^i\left(-\beta_i{\rm tr}F_1^2 + \beta_i{\rm tr}F_2^2 + f_{iAB}\mathcal{F}^A\mathcal{F}^B + \frac{1}{12}\gamma_i{\rm tr}R^2\right)\right] \; . \tag{5.14}$$

[4]Here and in the following, the index 10 refers to quantities measured in the 10–dimensional string metric, as opposed to the 11–dimensional Einstein metric used earlier.

From the reduction of the 10–dimensional Bianchi identity we have for the field strengths

$$H = dB - \frac{\alpha'}{2\sqrt{2}}\omega_3 \tag{5.15}$$

$$X^i = d\chi^i - \frac{\alpha'}{\sqrt{2}v^{1/3}}\eta^i_A \mathcal{A}^A \tag{5.16}$$

with the Chern–Simons form ω_3 satisfying

$$d\omega_3 = \mathrm{tr}F_1^2 + \mathrm{tr}F_2^2 + \sum_A (\mathcal{F}^A)^2 - \mathrm{tr}R^2 \ . \tag{5.17}$$

To simplify the notation, we have used indices $A, B, \cdots$ to denote the two E_8 factors as well as the various $U_I(1)$ factors. So, for example, $A = (na)$ where $n = 1, 2$ and a runs over the $U_I(1)$ factors in each sector. Correspondingly, we have defined the following short–hand notation

$$\eta^i_A = (\eta^i_{1a}, \eta^i_{2a}) \ , \qquad \hat{\eta}^i_A = (-\eta^i_{1a}, \eta^i_{2a}) \tag{5.18}$$

and

$$f_{iAB} = \begin{pmatrix} \frac{1}{6\pi^2} d_{ijk}\eta^j_{1a}\eta^k_{1b} - \beta_i\delta_{ab} & -\frac{1}{12\pi^2} d_{ijk}\eta^j_{1a}\eta^k_{2b} \\ -\frac{1}{12\pi^2} d_{ijk}\eta^j_{1a}\eta^k_{2b} & \frac{1}{6\pi^2} d_{ijk}\eta^j_{2a}\eta^k_{2b} + \beta_i\delta_{ab} \end{pmatrix} \tag{5.19}$$

for the various coupling constants in the effective action. The four–dimensional Newton constant κ_4 and the gauge coupling g_0 are defined as

$$\kappa_4^2 = \frac{\kappa_{10}^2}{v} \ , \qquad g_0^2 = \frac{2\kappa_{10}^2}{v\alpha'} \ . \tag{5.20}$$

It is useful to recall the structure of the gauge fields. The low–energy gauge group $\mathcal{G}_n$ in each sector is given by the product

$$\mathcal{G}_n = \mathcal{H}_n \times \mathcal{J}_n \ , \tag{5.21}$$

where $\mathcal{H}_n$ contains the semi–simple part and the $U_{II}(1)$ generators (which are not part of the internal structure group) while $\mathcal{J}_n$ contains the $U_I(1)$ generators (which are part of the internal structure group). Gauge fields in $\mathcal{H}_n$ are denoted by A_n with field strengths F_n. The $U_I(1)$ fields of type I are denoted by $\mathcal{A}^a_n$ with field strengths $\mathcal{F}^a_n$ or $\mathcal{A}^A$ and $\mathcal{F}^A$ in the above index convention. We also recall that the topological numbers β_i and γ_i, related to the second Chern characters of the internal bundles, have been defined in Eq. (2.13) and (2.14), respectively. Furthermore, d_{ijk} are the Calabi–Yau intersection numbers (2.20).

5.3 Classical variation of the action

We would now like to compute the classical gauge variation of the above action. We introduce the transformation parameters Λ^A for the $U_I(1)$ fields $\mathcal{A}^A$ and the transformation parameters Λ_n for the gauge fields A_n in $\mathcal{H}_n$. Furthermore, we will need the spin connection w and the associated transformation parameter Λ_L. We write explicitly

$$
\begin{aligned}
\delta \mathcal{A}^A &= d\Lambda_A && (5.22)\\
\delta A_n &= d\Lambda_n + A_n\Lambda_n - \Lambda_n A_n && (5.23)\\
\delta w &= d\Lambda_L + w\Lambda_L - \Lambda_L w\ . && (5.24)
\end{aligned}
$$

It is useful to note the resulting transformation of the Chern–Simons form w_3 defined in Eq. (5.17). It is given by

$$
\delta w_3 = \sum_A \Lambda^A \mathcal{F}^A + \sum_{n=1}^{2} d\,\mathrm{tr}(\Lambda_n dA_n) - d\,\mathrm{tr}(\Lambda_L dw)\ . \qquad (5.25)
$$

As before, the non–trivial gauge transformations of antisymmetric tensor fields are triggered by the Bianchi identities (5.15) and (5.16). Requiring the field strengths H and X^i to be gauge invariant, we find the transformation laws

$$
\begin{aligned}
\delta B &= \frac{\alpha'}{2\sqrt{2}}\left(\sum_A \Lambda^A \mathcal{F}^A + \sum_{n=1}^{2} \mathrm{tr}(\Lambda_n dA_n) - \mathrm{tr}(\Lambda_L dw)\right) && (5.26)\\
\delta \chi^i &= \frac{\alpha'}{\sqrt{2}}\eta^i_A \Lambda^A\ . && (5.27)
\end{aligned}
$$

This leads to the following classical variation of the action

$$
\begin{aligned}
\delta_{\mathrm{cl}} S_4 = -3\sqrt{2}\pi^2 k\alpha' \int_{M_4} \Bigg\{ \Lambda^A \Bigg[\eta^i_A \left(-\beta_i \mathrm{tr}F_1^2 + \beta_i \mathrm{tr}F_2^2 + \frac{1}{12}\gamma_i \mathrm{tr}R^2 \right) \\
+ \left(\eta^i_A f_{iBC} + \beta_i \hat{\eta}^i_C \delta_{AB}\right) \mathcal{F}^B \mathcal{F}^C \Bigg] \\
+ \beta_i \hat{\eta}^i_A \left[\mathrm{tr}(\Lambda_1 dA_1) + \mathrm{tr}(\Lambda_2 dA_2) - \mathrm{tr}(\Lambda_L dw)\right] \mathcal{F}^A \Bigg\}\ . \qquad (5.28)
\end{aligned}
$$

To understand the structure of this variation and to compare it to the previous results we should write it in a more explicit form and split all terms explicitly into the two sectors. Doing this, one arrives at the somewhat uncomfortable form

$$
\begin{aligned}
\delta_{\rm cl} S_4 \;=\; & -3\sqrt{2}\pi^2 k\alpha' \int_{M_4} \Bigg\{ \\
& \Lambda_1^a \Bigg[-\beta_i \eta_{1a}^i {\rm tr} F_1^2 + \beta_i \eta_{1a}^i {\rm tr} F_2^2 + \frac{1}{12}\gamma_i \eta_{1a}^i {\rm tr} R^2 && (5.29a) \\
& \quad + \left(-\beta_i \eta_{1a}^i \delta_{bc} - \beta_i \eta_{1c}^i \delta_{ab} + \frac{1}{6\pi^2} d_{ijk} \eta_{1a}^i \eta_{1b}^j \eta_{1c}^k \right) \mathcal{F}_1^b \mathcal{F}_1^c && (5.29b) \\
& \quad + \left(\beta_i \eta_{2c}^i \delta_{ab} - \frac{1}{6\pi^2} d_{ijk} \eta_{1a}^i \eta_{1b}^j \eta_{2c^k} \right) \mathcal{F}_1^b \mathcal{F}_2^c && (5.29c) \\
& \quad + \left(\beta_i \eta_{1a}^i \delta_{bc} + \frac{1}{6\pi^2} d_{ijk} \eta_{1a}^i \eta_{2b}^j \eta_{2c^k} \right) \mathcal{F}_2^b \mathcal{F}_2^c \Bigg] && (5.29d) \\
& \Lambda_2^a \Bigg[-\beta_i \eta_{2a}^i {\rm tr} F_1^2 + \beta_i \eta_{2a}^i {\rm tr} F_2^2 + \frac{1}{12}\gamma_i \eta_{2a}^i {\rm tr} R^2 && (5.29e) \\
& \quad + \left(+\beta_i \eta_{2a}^i \delta_{bc} - \beta_i \eta_{2c}^i \delta_{ab} + \frac{1}{6\pi^2} d_{ijk} \eta_{2a}^i \eta_{2b}^j \eta_{2c}^k \right) \mathcal{F}_2^b \mathcal{F}_2^c && (5.29f) \\
& \quad + \left(-\beta_i \eta_{2b}^i \delta_{ac} - \frac{1}{6\pi^2} d_{ijk} \eta_{2a}^i \eta_{1b}^j \eta_{2c^k} \right) \mathcal{F}_1^b \mathcal{F}_2^c && (5.29g) \\
& \quad + \left(-\beta_i \eta_{2a}^i \delta_{bc} + \frac{1}{6\pi^2} d_{ijk} \eta_{2a}^i \eta_{1b}^j \eta_{1c^k} \right) \mathcal{F}_1^b \mathcal{F}_1^c \Bigg] && (5.29h) \\
& + {\rm tr}(\Lambda_1 dA_1) \Big[-\beta_i \eta_{1a}^i \mathcal{F}_1^a + \eta_{2a}^i \beta_i \mathcal{F}_2^a \Big] && (5.29i) \\
& + {\rm tr}(\Lambda_2 dA_2) \Big[-\beta_i \eta_{1a}^i \mathcal{F}_1^a + \eta_{2a}^i \beta_i \mathcal{F}_2^a \Big] && (5.29j) \\
& + {\rm tr}(\Lambda_L dw) \Big[\beta_i \eta_{1a}^i \mathcal{F}_1^a - \eta_{2a}^i \beta_i \mathcal{F}_2^a \Big] \Bigg\} && (5.29k)
\end{aligned}
$$

for the classical variation.

5.4 Quantum variation of the action

The four–dimensional effective action that we have computed should be anomaly–free in the same way as the 10–dimensional action that it originates from is. As a consequence, the classical variation of the action computed above should cancel the quantum variation due to triangle diagrams. As a check we would like to verify this explicitly.

The triangle anomaly coefficients have already been generally defined in Eqs. (3.6a–3.6f). In fact, their specific form has to be exactly the same as for the five–dimensional heterotic M–theory, since they depend on the particle content in the gauge sectors only. We can, therefore, just use the result from section 3.2 which we repeat here for convenience

$$\begin{aligned}
\mathcal{C}_n &= 0 && (5.30)\\
\mathcal{C}_{na} &= \mp\frac{1}{8\pi}\eta^i_{na}\beta_i && (5.31)\\
\mathcal{C}_{nab} &= 0 && (5.32)\\
\mathcal{C}_{nabc} &= \frac{3}{8\pi}\left[\mp\beta_i\eta^i_{n(a}\delta_{bc)}+\frac{1}{12\pi^2}d_{ijk}\eta^i_{na}\eta^j_{nb}\eta^k_{nc}\right] && (5.33)\\
\mathcal{C}^{(L)}_n &= 0 && (5.34)\\
\mathcal{C}^{(L)}_{na} &= \frac{3}{2\pi}\left(\mp\beta_i-\frac{1}{12}\gamma_i\right)\eta^i_{na}\ . && (5.35)
\end{aligned}$$

The upper (lower) sign refers to the $n=1$ ($n=2$) sector. In five dimensions, the quantum variation was split into two parts, one on each orbifold plane. The four–dimensional quantum variation is, of course, just the sum of these two parts. From Eq. (3.8) it is given by

$$\delta_{\mathrm{Q}}S_4=\frac{1}{16\pi^2}\sum_{n=1}^{2}\int_{M_4}\Lambda^a_n\left[\mathcal{C}_{na}\mathrm{tr}F_n^2-\frac{1}{24}\mathcal{C}^{(L)}_{na}\mathrm{tr}R^2+\frac{1}{3}\mathcal{C}_{nabc}\mathcal{F}^b_n\mathcal{F}^c_n\right]\ . \tag{5.36}$$

Given that this quantum variation was cancelled by the classical variation of the five–dimensional action (3.5), how can it be cancelled by the significantly more complicated four–dimensional classical variation (5.29)? Again the answer is related to the ambiguity in the triangle anomaly. The classical variation (5.29) "wants" to cancel a quantum variation in a specific regularization. Comparing the four–dimensional classical variation (5.29) with its five–dimensional counterpart (3.5) there are two essential complications. First, in addition to the anomalous variations of the $U_I(1)$ fields that we had in five dimensions, the four–dimensional expression contains anomalous variations of the $\mathcal{H}_n$ parts of the gauge symmetries and gravity as well. Secondly, whereas in five dimensions the terms in the anomalous variation were naturally split between the two orbifold planes, the four–dimensional variation contains terms that mix the two sectors.

The first complication can be accounted for by regularizing the mixed $U_I(1)$ gauge and gravitational anomalies in a non–standard way. Usually, this is done such that the $U_I(1)$ is anomalous while the rest of the gauge group and gravity are anomaly free. Here we should adopt a mixed scheme in which all parts become anomalous. Rewriting the first two terms in the quantum variation (5.36) in such a way, they can be cancelled by the terms in (5.29a), (5.29e) and (5.29i–5.29k) in the classical variation (5.29). To check that this works, one has to verify that the coefficients of the corresponding terms in (5.29a), (5.29e) and (5.29i–5.29k) add up to what is required from the standard form of the anomaly (5.36). This is indeed the case. What about the cubic anomaly terms? Here we can use the same ambiguity and rewrite the last term in the quantum variation to be cancelled by the remaining terms in (5.29). In particular, in this way one can account for the terms in (5.29) that mix the two sectors. To see this, one should add up the coefficients of corresponding terms in (5.29b–5.29d) and (5.29f–5.29h). Doing this, we see that indeed all mixed coefficients cancel. This is essential, because a “real” anomaly that mixes the two sectors would be inconsistent. The two sectors are completely hidden with respect to one another so there are simply no fermions available that could generate a mixed sector anomaly. We finally conclude that four–dimensional classical and quantum variations indeed cancel as they should.

5.5 Discussion

What is the structure of four–dimensional anomaly cancellation from these results? All the essential information can be extracted by comparing the general definition of the anomaly coefficients (3.6a–3.6f) and their special form (5.30)–(5.35) along with the anomalous (quantum) variation which we can take to be in its standard form (5.36). It is clear, that this leads to exactly the same conclusions as in the case of heterotic M–theory in five dimensions.

Let us summarize the results from section 3.3 briefly. Recall that $\mathcal{H}_n$ consist of the non–abelian and the $U_{II}(1)$ gauge fields, collectively called A_n. Furthermore, we denote the $U_I(1)$ gauge fields by $\mathcal{A}_n^a$, where $a = 1, \ldots, k_n$ and the graviton is denoted by g. We then refer to a triangle diagram by specifying the triple of gauge fields to which it couples. Then the anomaly structure in *each* sector is as follows :

- The $\mathcal{H}_n$ part of the gauge group is anomaly–free in the sense that the A_n^3 and $A_n gg$ anomalies vanish.
- The mixed $\mathcal{A}_n^a \mathcal{A}_n^b A_n$ anomalies vanish.
- After a suitable choice of basis there is at most a single $U_I(1)$ gauge field, say $\mathcal{A}_n^1$ with all three remaining types of anomalies non–vanishing. That is, we may have anomalies of type $\mathcal{A}_n^1 A_n A_n$, $\mathcal{A}_n^1 \mathcal{A}_n^a \mathcal{A}_n^b$ and $\mathcal{A}_n^1 gg$. In terms of the topological data, the associated anomaly coefficients are given in Eq. (3.7b), (3.7d) and (3.7f). The remaining $U_I(1)$ factors are free of mixed gauge anomalies, that is the $\mathcal{A}_n^a A_n A_n$ anomaly vanishes for $a > 1$.
- The coefficient of $\mathcal{A}_n^1 A_n A_n$ is independent of which specific gauge field A_n within $\mathcal{H}_n$ is considered. Apart from this restriction, the three non–vanishing anomaly coefficients are generically unrelated.
- After another choice of basis there is at most one among the remaining $U_I(1)$ symmetries, say $\mathcal{A}_n^2$, with mixed gravitational and cubic anomaly. In other words, the anomalies of type $\mathcal{A}_n^2 gg$ and $\mathcal{A}_n^2 \mathcal{A}_n^a \mathcal{A}_n^b$ can be non–vanishing. Again the two associated anomaly coefficients are unrelated generically. All other $\mathcal{A}_n^a$ for $a > 2$ have cubic anomalies at most.

We have seen that the four– and five–dimensional pictures are completely consistent with one another. However, the essential structure is much easier seen in five dimensions where the two sectors of the theory are located on the orbifold planes and, hence, are "neatly" separated. What causes the complication in four dimensions? A reduction on the orbifold is not a simple truncation of all fields. In Kaluza-Klein terms, setting all the y-dependent modes equal to their vacuum expectation values (zero or other constant values) does not constitute a consistent truncation of the $D = 5$ theory down to $D = 4$.[5] Instead, what one has to do is to integrate out all the higher Kaluza-Klein fields of the bulk theory. This generates "interactions" across the orbifold and gives rise, among other things, to the more complicated structure of anomaly terms. In particular, the resulting $D = 4$ anomaly structure contains cross terms between the two E_8 sectors that from a purely $D = 4$ perspective would not appear to be necessary (*i.e.* they could be removed by appropriate regularization or renormalization), but which are unavoidable here as a result of the higher-dimensional anomaly structure.

5.6 Dualizing B

Usually, the four–dimensional two–form B is dualized to a scalar σ. While this does not effect the anomalous variation of the action under gauge symmetries, of course, it is nevertheless instructive to carry this out explicitly. We write $H_0 = dB$ and add the term

$$\frac{1}{\kappa_4^2}\int_{M_4} H_0 \wedge d\sigma \tag{5.37}$$

to the action (5.12)–(5.14). Then, integrating out H_0 we find

$$S_{4,\mathrm{kin}} = -\frac{1}{2\kappa_4^2}\int_{M_4}\left[\frac{1}{V^2}\Sigma\wedge *\Sigma + 2G_{ij}X^i\wedge *X^j\right] - \frac{1}{4g_0^2}\int_{M_4}\sqrt{-g}V\left[\mathrm{tr}F_1^2 + \mathrm{tr}F_2^2 + \sum_A(\mathcal{F}^A)^2\right] . \tag{5.38}$$

and

$$S_{4,\mathrm{top}} = \frac{\alpha'}{\sqrt{2}}\kappa_4^2\int_{M_4} w_3\wedge\Sigma - 6\pi^2 k\int_{M_4}\chi^i\left(-\beta_i\mathrm{tr}F_1^2 + \beta_i\mathrm{tr}F_2^2 + f_{iAB}\mathcal{F}^A\mathcal{F}^B + \frac{1}{12}\gamma_i\mathrm{tr}R^2\right) . \tag{5.39}$$

where the constants η^i_A, $\hat{\eta}^i_A$ and f_{iAB} have been defined in (5.18)–(5.19) and the field strengths Σ and X^i are given by

$$\Sigma = d\sigma + 12\pi^2 k\kappa_4^2\,\beta_i\hat{\eta}^i_A\mathcal{A}^A \tag{5.40}$$

$$X^i = d\chi^i - \frac{\alpha'}{\sqrt{2}v^{1/3}}\eta^i_A\mathcal{A}^A . \tag{5.41}$$

[5] The lack of Kaluza-Klein consistency in the $5 \to 4$ reduction, with consequent impact upon the $D = 4$ interaction structure, should be compared with the "essentially consistent" reduction from $D = 11$ to $D = 5$ on the Calabi-Yau manifold. In the latter case, integration of the higher Kaluza-Klein modes of the Calabi-Yau manifold does not produce corrections in the $D = 5$ effective action, at least at the level of terms containing no more than two derivatives [47].

Recall here that σ is the axionic part of the dilaton superfield S while χ^i are the axions in the T^i moduli. As usual, the two last equations can be used to deduce gauge transformation laws for the dilaton and the T moduli (for the T moduli the transformation has been given in Eq. (5.27)) by requiring the field strengths to be gauge invariant. We see that also the dilaton transforms non–trivialy, in general, as did its dual B. However, unlike for the T moduli, the dualized action (5.38), (5.39) is invariant under this transformation of the dilaton since it depends only on the field strength Σ. In particular, the universal coupling of the dilaton to the gauge fields is not given by σdw_3 but by Σw_3. Hence we see that the anomaly cancellation is essentially controlled by the transformation of the T moduli. This works thanks to the well–known T^i dependent threshold correction to the gauge kinetic function, of which the second part of Eq. (5.39) represents the imaginary part.

5.7 Relation to the usual picture

Finally, we should discuss how the above results relate to the usual picture of four–dimensional heterotic anomaly cancellation. To do this, it is useful to review the standard argument given in the literature [13]. One starts with the 10–dimensional Green–Schwarz term, *i.e.* roughly with an expression of the form

$$B \wedge F^4 \ . \tag{5.42}$$

Taking three of the gauge fields to be internal and the remaining one to be external, this leads to a four–dimensional term of the form

$$cB \wedge \mathbf{F} \tag{5.43}$$

where B is the four–dimensional two–form, $\mathbf{F} = d\mathbf{A}$ is a four–dimensional $U(1)$ gauge field strength and c is some coefficient. Dualizing B to the axion field σ, the above term becomes

$$c\partial_\mu \sigma \mathbf{A}^\mu \ . \tag{5.44}$$

In analogy with the previous subsection, this term shows that σ should be transformed under the gauge symmetry associated to $\mathbf{A}$. Due to the coupling σF^2 and σR^2 of the dilaton to all gauge fields and gravitation this then leads to an anomalous classical variation that should cancel the $U(1)$ triangle anomalies associated to $\mathbf{A}$. From this line of reasoning a number of properties of the quantum anomaly can be deduced. It should be universal, that is, it should be the same for all factors of the gauge group and for gravity. In particular, this includes the hidden and the observable gauge group. The reason for this property is the universal coupling of the dilaton. Furthermore, there is at most one such anomalous $U(1)$ symmetry. This is the standard reasoning.

There are two hidden assumptions in the above line of argument. First, it is assumed that only the dilaton, but not the T moduli, vary under the $\mathbf{A}$ gauge symmetry. Secondly, it is assumed that the coefficient c in (5.43) is non–zero. The first assumption necessarily implies that the $U(1)$ symmetry in question is of type II. Otherwise (if it were type I) the $U(1)$ would be part of the internal structure group. From the $\mathrm{tr}F^2$ terms in the (10–dimensional) Bianchi identity with one F external and the other internal, this would then lead to a term proportional to $\mathbf{A}$ in the T^i Bianchi identities. In other words, $\mathbf{A}$ would appear on the right hand side of Eq. (5.41) and hence the T^i would have to transform under the $U(1)$ symmetry. We have assumed that this is not the case and, hence, we are dealing with a type II symmetry. On the other hand, we have shown that the type II symmetries are always anomaly free. This must mean that the coefficient c in Eq. (5.43) is zero and, therefore, the second assumption above is violated. In fact, it is easy to understand why c vanishes for type II symmetries [17]. The only term in the anomaly polynomial (5.6) that could lead to a coupling (5.43) is the first term proportional to $\mathrm{tr}(F^4)$. This is because an expression of the form $\mathrm{tr}(\bar{F}\mathbf{F})$ involving the internal field strength $\bar{F}$ always vanishes if $\mathbf{F}$ is of type II. On the other hand, it is known that E_8 has no independent quartic invariant. As a consequence, the anomaly polynomial can be written in terms of second order invariants only, as it has been done in Eq. (5.7). This explains why c is always zero for $U_{II}(1)$ symmetries and, hence, why the coupling (5.43) does not exist in this case [6]. To summarize, we have found that one of the assumptions in the standard argument is not satisfied and, hence, that the conclusions about the structure of low–energy anomaly cancellation do not apply to the $E_8 \times E_8$ case. Essentially, the statement is that the conventional anomaly properties described above apply to an anomalous type II $U(1)$ symmetry. While for the $SO(32)$ heterotic theory type I as well as type II $U(1)$ symmetries can be anomalous, we have seen that type II symmetries are always anomaly–free in the $E_8 \times E_8$ case. Using the standard (type II) assumptions about anomalous $U(1)$ symmetries for $E_8 \times E_8$ means, therefore, applying them to an empty set. On the other hand, we have seen that type I $U(1)$ symmetries can be anomalous in the $E_8 \times E_8$ case and that their properties are quite different from the usual ones.

6 Phenomenological issues

6.1 Model building with anomalous $U(1)$ symmetries

From the perspective of low–energy model building it is an important question as to which anomalous $U(1)$ symmetries (with the quantum anomaly being cancelled by the Green–Schwarz mechanism) can be added to the MSSM (or extensions thereof) within the context of heterotic string– or M–theory effective actions. This question is answered by our results for the anomaly coefficients. We remind the reader, that, within each sector, we have split our low–energy gauge groups $\mathcal{G}_n$,

[6] This is different for the $SO(32)$ heterotic theory where there exists an independent fourth order invariant. In fact, the standard embedding leads to an example of an anomalous type II symmetry in this case [13].

where $n = 1, 2$ labels the observable and the hidden sectors, into two parts, namely

$$\mathcal{G}_n = \mathcal{H}_n \times \mathcal{J}_n \tag{6.1}$$

Here $\mathcal{H}_n$ contains the complete semi–simple part of the gauge group as well as the $U(1)$ symmetries of type II. The $\mathcal{J}_n$ part, on the other hand, contains the $U(1)$ factors of type I. The generators of $\mathcal{H}_n$ are generically called T_n, with the associated gauge fields A_n. The $\mathcal{J}_n$ generators are denoted by Q_n^a with gauge fields $\mathcal{A}_n^a$, where a runs over the various $U(1)$ factors in $\mathcal{J}_n$. In Eqs. (3.6a–3.6f) we have defined in general all possible anomaly coefficients between those two parts of the gauge group and also with gravity. Their specific forms within $E_8 \times E_8$ models, in terms of data related to the underlying compactification, has been given in (5.30)–(5.35). It is these latter equations that constitute the crucial input for anomalous $U(1)$ model building in the "bottom up" approach. Recall that, from those coefficients, the $\mathcal{H}_n$ parts of the gauge group are anomaly–free while the $\mathcal{J}_n$ contain the potentially anomalous $U(1)$ symmetries. The generic structure of the anomalous $U(1)$ symmetries in $\mathcal{J}_n$ that results has been discussed in section 5.5. Here we would like to comment on a few specific aspects conceivably relevant for model building.

A general feature of our structure is the independence of the observable and the hidden sector. For example, one may have two anomalous $U(1)$ symmetries, one in each sector, with generically unrelated anomaly coefficients. This could conceivably be important for relevant hidden sector physics such as supersymmetry breaking. We also gain some flexibility in the observable sector. The anomaly coefficients for the mixed gauge anomaly ($\mathcal{A}_n^a A_n A_n$ diagrams), the mixed gravitational anomaly ($\mathcal{A}_n^a gg$ diagrams where g is the graviton) and the cubic anomaly ($\mathcal{A}_n^a \mathcal{A}_n^b \mathcal{A}_n^c$ diagrams) are generically unrelated. There can also be more than one anomalous $U(1)$ symmetry in the observable sector.

A radical new feature would have been the possibility of having mixed $U(1)$ gauge anomaly coefficients ($\mathcal{A}_n^a A_n A_n$ diagrams) depending on the gauge group factor within $\mathcal{H}_n$. However, as we have shown these anomaly coefficients are always gauge–factor independent in heterotic string–theory or M–theory. In our more general five–dimensional approach, adopted in section 4, we have, however, found that such a possibility can be realized. Although the associated five–dimensional models are anomaly–free by construction, it is not yet clear whether or not they are part of M–theory. It is conceivable that they can be obtained in the context of heterotic M–theory vacua with five–branes [33, 36, 37, 38].

6.2 Special classes of compactifications

So far, we have discussed the generic structure of anomaly cancellation. This is the structure that arises without any further information about the details of the compactification. Within a given class of compactifications, however, the information can be much more precise. Technically speaking, for such a class of compactifications one typically has some additional information about the topological numbers that determine the anomaly coefficients via the relations (5.30)–(5.31). This in turn, restricts the structure of the anomaly coefficients and, hence, the structure of anomalous $U(1)$ symmetries within the given class.

As an illustration of this, let us present an example. A special role within heterotic M–theory is played by vacua for which the topological numbers β_i, defined in Eq. (2.13), vanish. Such vacua are also called symmetric. Recently, it has been shown [39] that such vacua based on Calabi–Yau three–folds indeed exist [7]. The characteristic property of low–energy theories based on symmetric vacua is the vanishing of the one–loop corrections to the four–dimensional effective action. In our context, we can use Eq. (5.30)–(5.35) to derive the special anomaly structure associated to symmetric vacua. We simply have to set $\beta_i = 0$ in those equations. As the most remarkable simplification, we then find from Eq. (5.31) that the mixed gauge anomalies ($\mathcal{A}_n^a A_n A_n$ diagrams) vanish in this case. That is, within effective theories originating from symmetric vacua, the $U_I(1)$ symmetries of type I may have at most mixed gravitational and cubic anomalies.

6.3 Fayet–Illiopolous terms

An important question in the context of anomalous $U(1)$ symmetries concerns the possible associated FI terms. The structure that we have found is sufficiently different from the usual one to reanalyze the question.

As usual, the FI terms can be read off from the $D = 4$, $\mathcal{N} = 1$ Kähler potential. The moduli superfields are defined in terms of their bosonic components as $S = V + i\sqrt{2}\sigma$ and $T^i = a_{10}^i + i\sqrt{2}\chi^i$. With these definitions, using the kinetic terms (5.38) along with Eq. (5.40) and (5.41), we find for the moduli Kähler potential K

$$\begin{aligned}\kappa_4^2 K &= -\ln\left[\frac{1}{6}d_{ijk}\left(T^i+\bar{T}^i-\frac{\alpha'}{v^{1/3}}\eta_A^i\mathcal{A}^A\right)\left(T^j+\bar{T}^j-\frac{\alpha'}{v^{1/3}}\eta_B^j\mathcal{A}^B\right)\left(T^k+\bar{T}^k-\frac{\alpha'}{v^{1/3}}\eta_C^k\mathcal{A}^C\right)\right]\\ &\quad -\ln\left(S+\bar{S}+\frac{\kappa_4^2c^3}{2\pi^2\alpha'}\mathcal{C}_A\mathcal{A}^A\right)\,. \end{aligned}\tag{6.2}$$

Recall that the indices $A = (na)$ run over the two sectors as well as over the various $U_I(1)$ symmetries in each sector. Let us first focus on the T part of this Kähler potential. Expanding to linear order in the gauge fields, we find that the FI terms from this part are proportional to

$$d_{ijk}\eta_A^i a^j a^k \tag{6.3}$$

where $a^i \sim a_{10}^i$ are the $(1,1)$ moduli. We have seen in Eq. (2.19), however, that the $(1,1)$ moduli

[7] It should be said that the examples of Ref. [39] do not directly apply to the present situation since they were restricted to Calabi–Yau spaces with $h^{1,1} = 1$. However, one expects symmetric vacua to exist for some Calabi–Yau spaces with $h^{1,1} > 1$.

are constrained in such a way that these expressions vanish [12]. Hence, we conclude that the T part of the Kähler potential does not lead to FI terms.

What about the dilaton part? We find

$$S_{\rm FI} = -\sum_{n=1}^{2} \int_{M_4} \sqrt{-g}\, \xi_{na} D_n^a \tag{6.4}$$

where the coefficients ξ_A are given by

$$\xi_{na} = \frac{c^3 M^2 g_0^2 ({\rm Re}(S))^{-1}}{16\pi^2} C_{na} \ . \tag{6.5}$$

Here M is the reduced Planck mass with $M^2 = \kappa_4^{-2}$. The anomaly coefficient C_{na} is related to the mixed $U(1)$ gauge anomaly ($\mathcal{A}_n^a A_n A_n$ diagrams). This coefficient has been generally defined in Eq. (3.6b) and its specific form has been given in Eq. (5.31). The result (6.5) is very reminiscent of the one usually given in the literature [13, 15, 16, 18]. However, there are some crucial differences. Usually, it is assumed that all anomaly coefficients have to be in fixed proportions to one another. The FI term can then be expressed in terms of either one of them. Here we have seen that the anomaly coefficients for the mixed gauge, the mixed gravitational and the cubic anomaly are, in fact, generally different. What we have found is that it is the mixed gauge anomaly coefficient that determines the size of the FI term. We have also seen that, after a suitable choice of basis, there can be one $U(1)$ symmetry with a mixed gauge anomaly per sector. This means that we potentially have two FI terms, one for the observable the other one for the hidden sector. In particular, one could have a situation with only a hidden sector FI term. The flexibility gained this way could conceivably be useful for some purposes such as D–term inflationary model building.

6.4 Threshold corrections

Another related question concerns the threshold corrections to the gauge kinetic function, particularly for the $U_I(1)$ fields. These can be read off from the action (5.14). The relevant part of the action can be written in the form

$$S_{\rm threshold} = -\frac{\epsilon_S}{4g_0^2} \int_{M_4} {\rm Im}(T^i) \left(-\beta_i {\rm tr} F_1^2 + \beta_i {\rm tr} F_2^2 + f_{iAB} \mathcal{F}^A \mathcal{F}^B \right) . \tag{6.6}$$

where the strong coupling expansion parameter ϵ_S is given by

$$\epsilon_S = \left(\frac{\kappa}{4\pi} \right)^{2/3} \frac{2c\pi^2 \rho}{v^{2/3}} \ . \tag{6.7}$$

This implies the usual result for the gauge kinetic functions of the $\mathcal{H}_n$ parts of the gauge group namely

$$f_n = S \mp \epsilon_S \beta_i T^i \ . \tag{6.8}$$

For the $U_I(1)$ fields of type I we find

$$f_{AB} = S\,\delta_{AB} + \epsilon_S f_{iAB} T^i \, . \tag{6.9}$$

The coefficients f_{iAB} were explicitly defined in Eq. (5.19). Note that the index $A = (na)$ runs over the two sectors as well as the various $U_I(1)$ symmetries. Generally, the coefficients f_{iAB} have off–diagonal pieces that mix type I $U_I(1)$ fields from the observable and the hidden sector.

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Appendix

A Calculation of anomaly coefficients

In this appendix, we would like to be more explicit about the calculation of the triangle anomaly coefficients (3.7a–3.7f) using standard technology [34, 35]. We start by listing some useful trace properties. For an $E_8 \times E_8$ gauge field F we have

$$\mathrm{Tr}(F^6) = \frac{1}{48}\mathrm{Tr}(F^2)\mathrm{Tr}(F^4) - \frac{1}{14400}\left(\mathrm{Tr}(F^2)\right)^3 \tag{A.1}$$

where Tr is the trace in the adjoint and we define as usual $\mathrm{tr} = \mathrm{Tr}/30$. Using a standard trick, we write $F = \alpha T + \beta \bar{F}$ where T and $\bar{F}$ are in $E_8 \times E_8$ and α and β are arbitrary coefficients. Inserting this into the above formula and extracting the $\alpha^3\beta^3$ term we find

$$\begin{aligned} \mathrm{Tr}(T^3\bar{F}^3) \;&=\; \frac{1}{240}\mathrm{Tr}(T\bar{F}^3)\mathrm{Tr}(T^2) + \frac{1}{240}\mathrm{Tr}(\bar{F}^2)\mathrm{Tr}(\bar{F}T^3) + \frac{1}{80}\mathrm{Tr}(T\bar{F})\mathrm{Tr}(T^2\bar{F}^2) \\ &\quad - \frac{1}{24000}\mathrm{Tr}(T^2)\mathrm{Tr}(T\bar{F})\mathrm{Tr}(\bar{F}^2) - \frac{1}{36000}\left(\mathrm{Tr}(T\bar{F})\right)^3 \, . \end{aligned} \tag{A.2}$$

Furthermore we need the E_8 relation

$$\mathrm{Tr}(F^4) = \frac{1}{100}\left(\mathrm{Tr}F^2\right)^2 \tag{A.3}$$

where F is now an E_8 gauge field. Inserting $F = \alpha T + \beta \bar{F}$, as above, where T and $\bar{F}$ are now in E_8 we find from the $\alpha\beta^3$ and the $\alpha^2\beta^2$ terms

$$\mathrm{Tr}(T\bar{F}^3) \;=\; \frac{1}{100}\mathrm{Tr}(T\bar{F})\mathrm{Tr}(\bar{F}^2) \tag{A.4}$$

$$\mathrm{Tr}(T^2\bar{F}^2) \;=\; \frac{1}{150}\left(\mathrm{Tr}(T\bar{F})\right)^2 + \frac{1}{300}\mathrm{Tr}(T^2)\mathrm{Tr}(\bar{F}^2) \, . \tag{A.5}$$

The above relations are useful to express the anomaly coefficients in terms of compactification data. To see this let us define the generic coefficient

$$\mathcal{C}(T) = \sum_r N_n^r \, \mathrm{tr}_{L_n^r}(T^3) \tag{A.6}$$

for some $E_8 \times E_8$ generator T. Using the index theorem (2.28) this can be put into the form

$$\mathcal{C}(T) = \frac{1}{6(2\pi)^3} \int_X \left[\mathrm{Tr}(T^3 \bar{F}^3) - \frac{1}{8} \mathrm{Tr}(T^3 \bar{F}) \mathrm{tr} \bar{R}^2 \right] , \tag{A.7}$$

where $\bar{F}$ and $\bar{R}$ are the internal Yang–Mills and gravity curvatures. We can rewrite this expression using Eq. (A.2). Then choosing the generator T to be part of one of the E_8 factors, that is $T = T_n$, and using Eqs. (A.4) and (A.5) we can simplify this further. Finally, with the definition of the topological numbers and the internal gauge fields given in section 2.2 we find for the anomaly coefficient

$$\mathcal{C}(T_n) = \frac{3}{8\pi} \left[\mp \mathrm{tr}(T_n^2) \mathrm{tr}(T_n Q_n^a) \eta_{na}^i \beta_i + \frac{1}{12\pi^2} \mathrm{tr}(T_n Q_n^a) \mathrm{tr}(T_n Q_n^b) \mathrm{tr}(T_n Q_n^c) \eta_{na}^i \eta_{nb}^j \eta_{nc}^k d_{ijk} \right] . \tag{A.8}$$

A similar calculation can be carried out for the coefficient relevant for the mixed gravitational anomaly defined by

$$\mathcal{C}^{(L)}(T_n) = \sum_r N_n^r \, \mathrm{tr}_{L_n^r}(T_n) \, . \tag{A.9}$$

Then, choosing T_n appropriately in these formulae, one derives the various anomaly coefficients (3.7a–3.7f).

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